Quantum Computing - Problem Set 4

Mateus de Oliveira Oliveira

School of Computer Science and Communication
KTH Royal Institute of Technology, 100-44, Stockholm, Sweden
mdeoliv@kth.se

1 Quantum Simulation - Based on [4]

Let G be a graph. The line graph of G, denoted by G^* is defined as follows: the vertices of G^* are the edges of G. Two vertices of G^* are connected if their corresponding edges in G are adjascent.

1. A crucial point in Markov and Shi's simulation result is the observation that if G has degree d then the treewidth of the line graph of G is at most d times the tree-width of G. Prove this fact (See the definition of treewidth on Lecture 9 - Slide 2).
2. Another crucial observation is the fact that the contraction complexity $CC(G)$ of G is equal to the tree-width of G^*. Show that fact. (See the definition of $CC(G)$ on Lecture 9 - Slide 10)
3. By a result of Robertson and Seymour, there is an algorithm that takes a graph G and outputs a tree decomposition of trewidth $O(tw(G))$ in time $V(G)^{O(1)}2^{O(tw(G))}$. Given this fact, the definition of the graph associated to a quantum circuit (Lect. 9 - Slide 5) and the definition of the tensor contraction (Slide 5), give a brief sketch of the proof that quantum circuits of logarithmic treewidth can be efficiently simulated in a classical computer.

2 The Local Hamiltonian Problem - Based on [3]

2.1 A

Explain the meaning of each term in the Hamiltoninan $H = H_{in} + H_{Prop} + H_{out}$ used in Kitaev's construction:

– Initialization Hamiltonian:

$$H_{in} = \left(\sum_{s=m+1} \Pi_s^{(1)} \right) \otimes |0\rangle\langle 0|$$

– Output Hamiltonian:

$$H_{out} = \Pi_1^0 \otimes |L\rangle\langle L|$$

– Propagation Hamiltonian $H_{prop} = \sum_{j=1}^{L} H_j$ where

$$H_j = -\frac{1}{2} U_j \otimes |j\rangle\langle j-1| - \frac{1}{2} U_j \otimes |j-1\rangle\langle j| + \frac{1}{2} I \otimes (|j\rangle\langle j| + |j-1\rangle\langle j-1|)$$

2.2 B

The Hamiltonian described above is not Local since besides acting on the two qubits needed to simulate the application of each unitary U_j (which is taken from some universal set of gates on two qubits), each term in H_{prop} must act on all $O(\log L)$ qubits of the clock register, where L is the number of gates in the circuit. One can solve this by replacing the binary clock register on $O(\log L)$ qubits with a unary clock register on L qubits where each state $|j\rangle$ is replaced by the state $|1, ..., 1, 0, ...0\rangle$ with j consecutive ones and $L - j$ zeroes. Now the Hamiltonians must be updated as follows, where $\Pi_1^{(b)}$ is the projector acting on the subspace of vectors in which the i-th qubit is equal to b:

- $|0\rangle\langle 0|$ is replaced by the projector Π_1^0.
- $|j\rangle|j\rangle$ is replaced by the projector $\Pi_j^{(1)} \otimes \Pi_{j+1}^{(0)}$.
- $|L\rangle\langle L|$ is replaced by $\Pi_L^{(1)}$.
- $|0\rangle\langle 1|$ is replaced by $(|0\rangle\langle 1|) \otimes \Pi_2^{(0)}$.
- $|j-1\rangle\langle j|$ is replaced by $\Pi_{j-1}^{(1)} \otimes (|0\rangle\langle 1|)_j \otimes \Pi_{j+1}^{(0)}$.
- $|L-1\rangle\langle L|$ is replaced by $\Pi_{L-1}^{(1)} \otimes (|0\rangle\langle 1|)_L$.

Explain the meaning of each of these replacements. The new obtained Hamiltonian is k-Local. What is the value of k?

3 Temperley Lieb Algebras and the Jones Polynomial - Based on [1]

1. Let B_n be the braid group on n strands. Then B_n is generated by $\{I, \sigma_1, ..., \sigma_{n-1}\}$ with relations $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i - j| \geq 1$ and $\sigma_{i+1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i$. Show that the graphical generators for B_4 satisfy these relations.
2. Let $TL_n(d)$ be the Temperley-Lieb algebra with generators $\{I, E_1, ..., E_{n-1}\}$ and relations $E_iE_j = E_jE_i$ for $|j - i| \geq 2$, $E_iE_{i+1}E_i = E_i$, $E_i^2 = E_i$. Verify that the Kauffman's graphical generators for $TL_4(d)$ (Lec 10 - Slide 9) satisfy these relations.
3. Let a be any complex number such that $d = -a^2 - a^{-2}$. Prove that $\rho(\sigma_i) = aE_i + a^{-1}I$ gives a representation of B_n on $TL_n(d)$.
4. Going further, it is possible to map each generator $E_i \in TL_n(d)$ to a unitary matrix $\tau(E_i)$ that can be represented by a circuit acting on $poly(n)$ qubits. Let a and d be complex numbers such that $d = -a^2 - a^{-2}$. The Jones polynomial of the trace closure B^{tr}(Lec 10. Slide 13) of a braid B at the point a is equal to

$$V_{B^{tr}}(a) = (-a)^{3w(B^{tr})} d^{n-1} tr(\tau(\rho_a(B))).$$

where w is the writle of a knot. Argue without going through many technicalities how you would use the Hadamard test to approximate $V_{B^{tr}}(a)$.

4 Non Local Games - Based on [5],[2]

In an entangled one-round two-prover protocol, the verifier sends a question s to Alice, a question t to Bob. Then Alice replies with a bit a and Bob replies with a bit b. The verifier decision is based only on the XOR of the answers. The goal of this question is to show that this kind of protocol can be simulated by using a unique quantum prover. Basically, the verifier sends a state $|\psi\rangle$ to the prover, the prover answer with a quantum state $|\phi\rangle$ to the verifier and the verifier decides to accept or reject. The simulation is as follows:

i) The verifier sends the state $|\psi\rangle = \frac{1}{\sqrt{2}}((-1)^a|0\rangle|0\rangle|s\rangle + (-1)^b|1\rangle|1\rangle|t\rangle)$ to the prover.

ii) The prover answers with a state $|\phi\rangle$.

iii) The verifier measures the quibits of the anwer according to the following projective measurement:
   - $P_0 = |\varphi^+\rangle\langle\varphi^+| \otimes I$
   - $P_1 = |\varphi^-\rangle\langle\varphi^-| \otimes I$
   - $P_{rej} = I - P_0 - P_1$ where
   -

$$|\varphi^+\rangle = \frac{|0\rangle|0\rangle|s\rangle + |1\rangle|1\rangle|t\rangle}{\sqrt{2}} \quad \text{and} \quad |\varphi^-\rangle = \frac{|0\rangle|0\rangle|s\rangle - |1\rangle|1\rangle|t\rangle}{\sqrt{2}}$$

iv) If the outcome is *reject*, then the verifier rejects immediately and concludes that the prover is cheating.

v) If the verifier receives the outcome $c \in \{0, 1\}$, then she considers that $c = a \oplus b$ is the XOR of the answers of the two "virtual" entangled provers.

Question 1: Show that if the provers are honest, they can always answer with the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}((-1)^a|0\rangle|0\rangle|s\rangle + (-1)^b|1\rangle|1\rangle|t\rangle)$$

and the verifier will accept the answer with the same probability as in the original protocol.

Question 2: It can be shown that if a prover is dishonest then the probability with which the verifier will accept is given by

$$\frac{1}{2}\sum_{s,t,c} \pi(s,t)V(s,t|c)\langle\alpha_s|\alpha_t\rangle$$

for unit vectors $|\alpha_s\rangle$ and $|\alpha_t\rangle$. Argue that by an application of Tsirelson's theorem (Lec 12 - Slide 10), this implies that the quantum prover cannot convince the verifier with higuer probability than the original two entangled provers whose answers are classical bits.

References

[1] Aharonov, Jones, and Landau. A polynomial quantum algorithm for approximating the jones polynomial. In *STOC: ACM Symposium on Theory of Computing (STOC)*, 2006.

[2] Cleve, Hoyer, Toner, and Watrous. Consequences and limits of nonlocal strategies. In *Annual IEEE Conference on Computational Complexity (formerly Annual Conference on Structure in Complexity Theory)*, volume 19, 2004.

[3] A. Y. Kitaev, A. H. Shen, and M. N. Vyalyi. *Classical and Quantum Computation.* American Mathematical Society, Boston, MA, USA, 2002.

[4] I. L. Markov and Y. Shi. Simulating quantum computation by contracting tensor networks. *SIAM Journal on Computing*, 38(3):963–981, 2008.

[5] S. Wehner. Entanglement in interactive proof systems with binary answers. In B. Durand and W. Thomas, editors, *STACS*, volume 3884 of *Lecture Notes in Computer Science*, pages 162–171. Springer, 2006.