A Hilbert style proof system for LTL

The meaning of individual axioms. Completeness

Preliminaries on proof systems

A proof system - a formal grammar definition of a sublanguage in the logic.

A proof system is

- sound, if it produces only valid formulas
- complete, if it produces all the valid formulas

We are only interested in sound proof systems.

Typically a proof system consists of

- axioms - concrete valid formulas
- proof rules - to derive valid formulas from other valid formulas

A Hilbert-style proof system for classical propositional logic

$\neg p \leftharpoondown\!\!\!\!\!\rightharpoonup p \Rightarrow \bot$

$$p \Rightarrow (q \Rightarrow p)$$

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$$

$$p \Rightarrow (q \Rightarrow (p \wedge q))$$

$$p \wedge q \Rightarrow p, \qquad p \wedge q \Rightarrow q$$

$$p \Rightarrow p \vee q, \qquad q \Rightarrow p \vee q$$

$$(p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$$

$$\neg\neg p \Rightarrow p$$

the rule Modus Ponens and the substitutivity rule

$$(MP) \qquad \frac{\varphi \quad \varphi \Rightarrow \psi}{\psi} \qquad\qquad (Sub) \qquad \frac{\varphi}{[\psi/p]\varphi}$$

Modus Ponens

$$(MP) \qquad \frac{\varphi \quad \varphi \Rightarrow \psi}{\psi}$$

MP is not terribly convenient in proof search. Must guess a lemma φ.

Exercise 1 Prove $p \Rightarrow p$ in the above system.

The substitutivity rule

$$(Sub) \qquad \frac{\varphi}{[\psi/p]\varphi}$$

Sub derives substitution instances of valid formulas. Alternative to Sub: use axiom schemata such as

$$\varphi \Rightarrow (\psi \Rightarrow \varphi)$$

$\dots$

Proofs and theorems in an arbitrary proof system S

A proof in S - a sequence of formulas $\varphi_1, \ldots, \varphi_n$,

each φ_i being either:

(a substitution instance of) an axiom of S

or

derived by a rule of S from formulas among $\varphi_1, \ldots, \varphi_{i-1}$

$\vdash_S \varphi$, if φ appears (in the end of) some proof in S

Proofs from given premises $\psi_1, \ldots, \psi_m$:

φ_i may be one of $\psi_1, \ldots, \psi_m$ as well

$\psi_1, \ldots, \psi_m \vdash_S \varphi$

A Hilbert-style proof system for K

Minimal normal uni-modal logic $\mathbf{K}$:

$\varphi ::= \bot \mid p \mid (\varphi \Rightarrow \varphi) \mid \Diamond\varphi \mid \Box\varphi$

$\langle W, R, V\rangle$, $W \neq \emptyset$, $R \subseteq W \times W$, $V : W \to \mathcal{P}(\mathbf{L})$

$M, w \models \Diamond\varphi$ if $M, w' \models \varphi$ for some $w' \in R(w)$, $\qquad \Box\varphi \rightleftharpoons \neg\Diamond\neg\varphi$

Any sound and complete proof system for classical propositional logic $+$

$$(\mathbf{K}) \qquad \Box(p \Rightarrow q) \Rightarrow (\Box p \Rightarrow \Box q) \qquad (N) \qquad \frac{\varphi}{\Box\varphi}$$

The subset of LTL with just $\circ$ and $\Diamond$ is the logic of the bi-modal frame

$$\langle \omega, \prec, \leq \rangle$$

where $\circ$ is $\Diamond_{\prec}$ and $\Diamond$ is $\Diamond_{\leq}$.

A proof system for LTL

$(A0)$		all classical propositional tautologies
$(A1)$	$(K_\Box)$	$\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$
$(A2)$	(Fun)	$\neg \circ \varphi \Leftrightarrow \circ\neg\varphi$
$(A3)$	$(K_\circ)$	$\circ(\varphi \Rightarrow \psi) \Rightarrow (\circ\varphi \Rightarrow \circ\psi)$
$(A4)$		$\Box(\varphi \Rightarrow \circ\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$
$(A5)$		$(\varphi \mathsf{U} \psi) \Leftrightarrow \psi \vee (\varphi \wedge \circ(\varphi \mathsf{U} \psi))$
$(A6)$		$(\varphi \mathsf{U} \psi) \Rightarrow \Diamond\psi$
(MP)		$\dfrac{\varphi \qquad \varphi \Rightarrow \psi}{\psi}$
$(N_\Box)$		$\dfrac{\varphi}{\vdash \Box\varphi}$
$(N_\circ)$		$\dfrac{\varphi}{\vdash \circ\varphi}$

The assignment method

$F = \langle W, R \rangle$, $F \models \alpha$, if $\langle W, R, V \rangle, w \models \alpha$ for all V, w

$F \models \alpha$ is defines a property of R. It is equivalent to

$F \models \forall P_1 \ldots \forall P_k \forall w \mathsf{ST}(\alpha)$ in second order predicate logic where

$$\mathsf{ST}(\bot) \rightleftharpoons \bot$$

$$\mathsf{ST}(p_i) \rightleftharpoons P_i(w)$$

$$\mathsf{ST}(\alpha_1 \Rightarrow \alpha_2) \rightleftharpoons \mathsf{ST}(\alpha_1) \Rightarrow \mathsf{ST}(\alpha_2)$$

$$\mathsf{ST}(\Diamond\alpha) \rightleftharpoons \exists v(R(w,v) \wedge [v/w]\mathsf{ST}(\alpha))$$

The assignment method

To prove $\forall P_1 \ldots \forall P_k \forall w \mathsf{ST}(\alpha) \Leftrightarrow \beta$ for a first order sentence β, one proves

$$\beta \Rightarrow \mathsf{ST}(\alpha) \text{ in f.o. logic and } \neg\beta \Rightarrow \exists w \exists P_1 \ldots \exists P_k \neg \mathsf{ST}(\alpha)$$

for suitable assignments to $P_1, \ldots, P_k$, which is the same as proving

$F \not\models \beta \to \langle W, R, V\rangle, w \models \neg\alpha$ for suitable w and $[\![p_i]\!] = \{w \in W : p_i \in V(w)\}$

$\sigma : \omega \to \mathcal{P}(\mathbf{L})$ can be viewed as a bi-modal model $\langle \omega, \prec, \leq, \sigma\rangle$.

Then $\langle \omega, \prec, \leq\rangle \models \alpha$ corresponds to a connection between $\prec$ and $\leq$.

The meaning of some axioms

Proposition 1 (Fun) Let $F = \langle W, R\rangle$ be a frame, $\circ = \Diamond_R$. Then
$F \models \neg \circ p \Leftrightarrow \circ\neg p$ iff R is a total function. ($[\![p]\!]_F = \emptyset, \{w_1\} \subset R(w_0)$)

Proposition 2 Let $F = \langle W, R_1, R_2\rangle$ be a frame, $\circ = \Diamond_{R_1} = \Box_{R_1}$, $\Diamond = \Diamond_{R_2}$. Then

$$F \models \Box p \Rightarrow \Box \circ p \text{ iff } R_2 \circ R_1 \subseteq R_2 \qquad ([\![p]\!]_F = R_1(w_0))$$

$$F \models \Box p \Rightarrow p \text{ iff } Id_W \subseteq R_2 \qquad ([\![p]\!]_F = W \setminus \{w_0\})$$

$R_2 \circ R_1 \subseteq R_2$ and $Id_W \subseteq R_2$ imply $R_1^* \subseteq R_2$

Proposition 3 ($A4$) Let $R_2 \circ R_1 \subseteq R_2$. Then

$$F \models \Box(p \Rightarrow \circ p) \Rightarrow (p \Rightarrow \Box p) \text{ iff } R_2 \subseteq R_1^* \qquad ([\![p]\!]_F = R_1^*(w_0))$$

The proof system for LTL again

$(A0)$		all classical propositional tautologies
$(A1)$	$(K_\Box)$	$\Box(\varphi \Rightarrow \psi) \Rightarrow (\Box\varphi \Rightarrow \Box\psi)$
$(A2)$	(Fun)	$\neg \circ \varphi \Leftrightarrow \circ\neg\varphi$
$(A3)$	$(K_\circ)$	$\circ(\varphi \Rightarrow \psi) \Rightarrow (\circ\varphi \Rightarrow \circ\psi)$
$(A4)$		$\Box(\varphi \Rightarrow \circ\varphi) \Rightarrow (\varphi \Rightarrow \Box\varphi)$
$(A5)$		$(\varphi \mathsf{U} \psi) \Leftrightarrow \psi \vee (\varphi \wedge \circ(\varphi \mathsf{U} \psi))$
$(A6)$		$(\varphi \mathsf{U} \psi) \Rightarrow \Diamond\psi$
(MP)		$\dfrac{\varphi \qquad \varphi \Rightarrow \psi}{\psi}$
$(N_\Box)$		$\dfrac{\varphi}{\vdash \Box\varphi}$
$(N_\circ)$		$\dfrac{\varphi}{\vdash \circ\varphi}$

Some useful admissible rules and theorems in LTL

A proof rule is admissible in a proof system, if it does not contribute new theorems.

Fact 1 If $\varphi_1, \ldots, \varphi_n \vdash_S \psi$, then $\dfrac{\varphi_1, \ldots, \varphi_n}{\psi}$ is an admissible rule.

Proposition 4 The rules below are admissible in LTL for $\mathsf{L} \in \{\Box, \circ\}$:

$$(Mono_{\mathsf{L}}) \qquad \frac{\varphi \Rightarrow \psi}{\mathsf{L}\varphi \Rightarrow \mathsf{L}\psi} \qquad (E_{\mathsf{L}}) \qquad \frac{\varphi \Leftrightarrow \psi}{\mathsf{L}\varphi \Leftrightarrow \mathsf{L}\psi}$$

Proof: Exercise. $\dashv$

Theorem 1 (syntactical form of replacement of equivalents)

If $\vdash_{LTL} \varphi \Leftrightarrow \psi$, then $\vdash_{LTL} [\varphi/p]\chi \Leftrightarrow [\psi/p]\chi$.

Proof: Induction on the construction of χ. Use $E_\circ$ for $\chi \doteq \circ\theta$. $\dashv$

For $\chi \doteq (\theta_1 \mathsf{U} \theta_2)$, let $\theta_i' \rightleftharpoons [\varphi/p]\theta_i$ and $\theta_i'' \rightleftharpoons [\psi/p]\theta_i$, $i = 1, 2$

1	$\theta'_i \Leftrightarrow \theta''_i$	$i = 1, 2,$ ind. hypothesis
2	$(\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow$	
	$(\theta'_2 \vee (\theta'_1 \wedge \circ(\theta'_1 \mathsf{U} \theta'_2)) \wedge \neg(\theta''_2 \vee (\theta''_1 \wedge \circ(\theta''_1 \mathsf{U} \theta''_2)))$	$A5$
3	$(\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow \circ(\theta'_1 \mathsf{U} \theta'_2) \wedge \neg \circ (\theta''_1 \mathsf{U} \theta''_2)$	$1, 2$
4	$(\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow \circ((\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2))$	$3,$ exercises
5	$\Box((\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow \circ((\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2)))$	$4, N_\Box$
6	$(\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow \Box((\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2))$	$5, A4$
7	$\Box((\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2)) \Rightarrow \Box\neg(\theta''_1 \mathsf{U} \theta''_2)$	$K_\Box$
8	$\neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow \neg\theta''_2$	$A5$
9	$\Box((\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2)) \Rightarrow \Box\neg\theta''_2$	$7, 8, N_\Box, K_\Box$
10	$(\theta'_1 \mathsf{U} \theta'_2) \Rightarrow \Diamond\theta'_2$	$A6$
11	$(\theta'_1 \mathsf{U} \theta'_2) \wedge \neg(\theta''_1 \mathsf{U} \theta''_2) \Rightarrow \Box\neg\theta''_2 \wedge \Diamond\theta'_2$	$6, 9, 10$
12	$\Box\neg\theta''_2 \wedge \Diamond\theta'_2 \Rightarrow \bot$	$1, N_\Box, K_\Box$
13	$(\theta'_1 \mathsf{U} \theta'_2) \Rightarrow (\theta''_1 \mathsf{U} \theta''_2)$	$11, 12$

Some useful theorems

(T1) $\Diamond\varphi \Leftrightarrow (\top \mathsf{U} \varphi)$

1	$\Box(\circ(\top\mathsf{U}\varphi) \Rightarrow (\top\mathsf{U}\varphi))$	$A5, N_\Box$
2	$\Box(\neg(\top\mathsf{U}\varphi) \Rightarrow \circ\neg(\top\mathsf{U}\varphi)) \Rightarrow (\neg(\top\mathsf{U}\varphi) \Rightarrow \Box\neg(\top\mathsf{U}\varphi))$	$A4$
3	$\circ\neg(\top\mathsf{U}\varphi) \Leftrightarrow \neg\circ(\top\mathsf{U}\varphi)$	$A2$
4	$\Diamond(\top\mathsf{U}\varphi) \wedge \Box(\circ(\top\mathsf{U}\varphi) \Rightarrow (\top\mathsf{U}\varphi)) \Rightarrow (\top\mathsf{U}\varphi)$	$2, 3$
5	$\Diamond(\top\mathsf{U}\varphi) \Rightarrow (\top\mathsf{U}\varphi)$	$1, 4$
6	$\Box(\neg(\top\mathsf{U}\varphi) \Rightarrow \neg\varphi)$	$A5, N_\Box$
7	$\Box\neg(\top\mathsf{U}\varphi) \Rightarrow \Box\neg\varphi$	$6, K_\Box, MP$
8	$\Diamond\varphi \Rightarrow \Diamond(\top\mathsf{U}\varphi)$	7
9	$\Diamond\varphi \Rightarrow (\top\mathsf{U}\varphi)$	$5, 8$
10	$\Diamond\varphi \Leftrightarrow (\top\mathsf{U}\varphi)$	$A6, 9$

$(T2)$ $\Box\varphi \Leftrightarrow \varphi \wedge \circ\Box\varphi$

1	$\Box\varphi \Leftrightarrow \neg(\top \mathsf{U} \neg\varphi)$	$T1$
2	$\neg(\top \mathsf{U} \neg\varphi) \Leftrightarrow \varphi \wedge \circ\neg(\top \mathsf{U} \neg\varphi)$	$A5$
3	$\circ\neg(\top \mathsf{U} \neg\varphi) \Leftrightarrow \circ\Box\varphi$	$1, E_\circ$
4	$\Box\varphi \Leftrightarrow \varphi \wedge \circ\Box\varphi$	$1, 2, 3$

$(T3)$ $\Box\varphi \Rightarrow \Box \circ \varphi$

1	$\Box\varphi \Rightarrow \varphi$	$T2$
2	$\circ\Box\varphi \Rightarrow \circ\varphi$	$1, Mono_\circ$
3	$\Box\varphi \Rightarrow \circ\Box\varphi$	$T2$
4	$\Box\varphi \Rightarrow \circ\varphi$	$2, 3$
5	$\Box\Box\varphi \Rightarrow \Box \circ \varphi$	$4, Mono_\Box$
6	$\Box(\Box\varphi \Rightarrow \circ\Box\varphi)$	$3, N_\Box$
7	$\Box(\Box\varphi \Rightarrow \circ\Box\varphi) \Rightarrow (\Box\varphi \Rightarrow \Box\Box\varphi)$	$A4$
8	$\Box\varphi \Rightarrow \Box \circ \varphi$	$6, 7, 5$

Exercises

Exercise 2 Find proofs for the following formulas

$$\circ(\varphi \wedge \psi) \Leftrightarrow \circ\varphi \wedge \circ\psi \tag{1}$$

$$\circ(\varphi \vee \psi) \Leftrightarrow \circ\varphi \vee \circ\psi \tag{2}$$

$$\Box(\varphi \Rightarrow \varphi') \Rightarrow ((\varphi \mathsf{U} \psi) \Rightarrow (\varphi' \mathsf{U} \psi)) \tag{3}$$

$$\Box(\psi \Rightarrow \psi') \Rightarrow ((\varphi \mathsf{U} \psi) \Rightarrow (\varphi \mathsf{U} \psi')) \tag{4}$$

$$(\varphi_1 \wedge \varphi_2 \mathsf{U} \psi) \Leftrightarrow (\varphi_1 \mathsf{U} \psi) \wedge (\varphi_2 \mathsf{U} \psi) \tag{5}$$

$$(\varphi \mathsf{U} \psi_1 \vee \psi_2) \Leftrightarrow (\varphi \mathsf{U} \psi_1) \vee (\varphi \mathsf{U} \psi_2) \tag{6}$$

Completeness of the LTL proof system

We prove that if $\not\vdash_{LTL} \neg\varphi$, then φ is satisfiable at a linear model.

Maximal consistent sets of formulas

Fix $\mathbf{L}$. Γ - a set of formulas in $\mathbf{L}$.

$LTL_{\mathbf{L}}$ - the set of the theorems of LTL written in $\mathbf{L}$.

$\mathrm{Cn}(\Gamma) = \{\psi \in \mathbf{L} : \Gamma \cup LTL_{\mathbf{L}} \vdash_{MP} \psi\}$

Lemma 1 $\mathrm{Cn}(\Gamma \cup \{\varphi\}) = \{\psi : \varphi \Rightarrow \psi \in \mathrm{Cn}(\Gamma)\}$

Definition 1 Γ is consistent, if $\bot \notin \mathrm{Cn}(\Gamma)$. φ is consistent if $\{\varphi\}$ is consistent.

Maximal consistent sets

Definition 2 Γ is maximal consistent, if it is not a proper subset of any other consistent set of formulas in $\mathbf{L}$.

Proposition 5 Let Γ be consistent and $\varphi \in \mathbf{L}$. Then either $\Gamma \cup \{\varphi\}$ is consistent, or $\Gamma \cup \{\neg\varphi\}$ is consistent, or both.

Corollary 1 Let Γ be maximal consistent and $\varphi \in \mathbf{L}$. Then either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$.

Proposition 6 (Lindenbaum lemma) Let Γ be consistent. Then there exists a maximal consistent set Γ' in $\mathbf{L}$ such that $\Gamma \subseteq \Gamma'$.

Proof of the Lindenbaum lemma

Let all the formulas in $\mathbf{L}$ occur in the sequence φ_i, $i < \omega$.

We construct a sequence of sets of formulas $\Gamma_0 = \Gamma \subseteq \Gamma_1 \subseteq \ldots \subseteq \Gamma_i \subseteq \ldots$:

$$\Gamma_{i+1} = \begin{cases} \Gamma_i \cup \{\varphi_i\}, & \text{if } \Gamma_i \cup \{\varphi_i\} \text{ is consistent} \\ \Gamma_i \cup \{\neg\varphi_i\}, & \text{otherwise.} \end{cases}$$

Γ_i is consistent for all i. Let $\Gamma' = \bigcup_{i<\omega} \Gamma_i$.

If $\bot \in \mathrm{Cn}(\Gamma')$, then there exist formulas $\psi_1, \ldots, \psi_k \in \Gamma'$, s.t. $LTL_{\mathbf{L}}, \psi_1, \ldots, \psi_k \vdash_{MP} \bot$, whence $\bot \in \mathrm{Cn}(\Gamma_n)$ for n s.t. $\psi_1, \ldots, \psi_k \in \Gamma_n$, which is a contradiction. Hence Γ' is consistent.

Let $\varphi \notin \Gamma'$. If $\varphi = \varphi_i$, then $\neg\varphi \in \Gamma_{i+1}$, whence $\bot \in \mathrm{Cn}(\Gamma_{i+1} \cup \{\varphi\}) \subseteq \mathrm{Cn}(\Gamma' \cup \{\varphi\})$. Hence Γ' is maximal consistent.

$\circ^{-1}\Gamma$

Γ - a set of formulas in $\mathbf{L}$; $\circ^{-1}\Gamma = \{\varphi : \circ\varphi \in \Gamma\}$.

Lemma 2 If $LTL_{\mathbf{L}}, \psi_1, \ldots, \psi_n \vdash_{MP} \chi$, then $LTL_{\mathbf{L}}, \circ\psi_1, \ldots, \circ\psi_n \vdash_{MP} \circ\chi$.

Proof: Induction on the length of the proofs of $LTL_{\mathbf{L}}, \psi_1, \ldots, \psi_n \vdash_{MP} \chi$, using $\mathbf{K}_\circ$. $\dashv$

Proposition 7 If Γ is consistent, then $\circ^{-1}\Gamma$ is consistent too.

Proof: If $\bot \in \mathrm{Cn}(\circ^{-1}\Gamma)$, then $\circ\bot \in \mathrm{Cn}(\Gamma)$. Besides, $\vdash_{LTL} \circ\bot \Rightarrow \bot$:

1	$\circ\neg\bot$	$N_\circ$
2	$\circ\neg\bot \Rightarrow \neg\circ\bot$	$A2$
3	$\circ\bot \Rightarrow \bot$	$1, 2$

$\dashv$

A model for an arbitrary given consistent φ

We construct a Kripke model $M = \langle W, R, I, V \rangle$ first.

Then we identify a behaviour s in M s.t. $\sigma_s, 0 \models \varphi$

$W = \{\Gamma \cap \mathrm{Cl}(\varphi) : \Gamma$ is maximal consistent$\}$

Since φ is consistent, there is an $w_0 \in w$ s.t. $\varphi \in w$.

$I = \{w_0\}$

$w' R w'' \leftrightarrow \circ^{-1} w' \subseteq w''$

$V(w) = \mathbf{L} \cap w$ - the prop. variables appearing in w as atomic formulas

Proposition 8 M is a model, i.e., R is serial, and M is finite.

A model for an arbitrary given consistent φ

$W = \{\Gamma \cap \mathrm{Cl}(\varphi) : \Gamma$ is maximal consistent$\}$

$I = \{w_0\}$, $\varphi \in w_0$, $w' R w'' \leftrightarrow \circ^{-1} w' \subseteq w''$, $V(w) = \mathbf{L} \cap w$

We want to identify a behaviour s in M s.t.

$s, i \models \psi$ iff $\psi \in s_i$ for $\psi \in \mathrm{Cl}(\varphi)$ and $s, 0 \models \varphi$ (because $s_0 = w_0$)

This holds for s s.t. $(\psi \mathsf{U} \chi) \in s_i$ entails $\chi \in s_{i+j}$ for some j.

Therefore,

given any behaviour prefix $s_0 \ldots s_n$ and $(\psi \mathsf{U} \chi) \in s_i$ for some $i \leq n$,

we want to be able to extend it to an $s_0 \ldots s_n s_{n+1} \ldots s_m$ s.t. $\chi \in s_m$.

$\widehat{w}$

x - a finite set of formulas

$\widehat{x} \rightleftharpoons \bigwedge x$ - the conjunction of all the formulas in x

Lemma 3 If $x \subset \mathrm{Cl}(\varphi)$, then $\quad \vdash_{LTL} \widehat{x} \Rightarrow \bigvee_{w \in W, w \supseteq x} \widehat{w}$.

Proof: We can assume that x is consistent. If $\psi, \neg\psi \in \mathrm{Cl}(\varphi) \setminus x$, then

$$\widehat{x} \Leftrightarrow \widehat{x \cup \{\psi\}} \vee \widehat{x \cup \{\neg\psi\}}.$$

$\dashv$

$\widehat{w} \rightleftharpoons \bigwedge w$ continued

Lemma 4 If $w \in W$, then $\quad \vdash_{LTL} \widehat{w} \Rightarrow \circ \left(\bigvee_{wRw'} \widehat{w'} \right)$

Proof: wRw' is equivalent to $\circ^{-1}w \subseteq w'$. By the previous lemma

$$\vdash_{LTL} \widehat{\circ^{-1}w} \Rightarrow \bigvee_{wRw'} \widehat{w'}.$$

Now, by $N_\circ$ and $\mathbf{K}_\circ$,

$$\vdash_{LTL} \circ \left(\widehat{\circ^{-1}w} \right) \Rightarrow \circ \bigvee_{wRw'} \widehat{w'}.$$

Since $\{\circ\psi : \psi \in \circ^{-1}w\} \subset w$, by the distributivity of $\circ$ over $\wedge$,

$$\vdash_{LTL} \widehat{w} \Rightarrow \circ \left(\widehat{\circ^{-1}w} \right).$$

$\dashv$

$\widehat{w} \rightleftharpoons \bigwedge w$ continued

Lemma 5 If $w \in W$, then $\qquad \vdash_{LTL} \widehat{w} \Rightarrow \Box\left(\bigvee_{wR^*w'} \widehat{w'}\right).$

Corollary 2 If $(\psi \mathsf{U} \chi) \in w$, then there is a $w' \in R^*(w)$ s.t. $\chi \in w'$.

Proof: Let θ be propositionally equivalent to $\neg\chi$, and $\theta \in \mathrm{Cl}(\varphi)$. Assume that

$$\theta \in w' \text{ for all } w' \in R^*(w)$$

for the sake of contradiction. Then, by the lemma and $\mathbf{K}_\Box$,

$$\vdash_{LTL} \widehat{w} \Rightarrow \Box\theta, \text{ i.e., } \vdash_{LTL} \widehat{w} \Rightarrow \Box\neg\chi.$$

This contradicts the consistency of w, because of the instance of $A6$

$$(\psi \mathsf{U} \chi) \Rightarrow \Diamond\chi.$$

$\dashv$

1	$\widehat{w'} \Rightarrow \circ\left(\bigvee_{w'Rw''} \widehat{w''}\right)$	previous lemma
2	$\bigvee_{wR^*w'} \widehat{w'} \Rightarrow \bigvee_{wR^*w'} \circ\left(\bigvee_{w'Rw''} \widehat{w''}\right)$	1 for every $w' \in R^*(w)$
3	$\bigvee_{wR^*w'} \widehat{w'} \Rightarrow \circ\left(\bigvee_{wR^*w'} \bigvee_{w'Rw''} \widehat{w''}\right)$	$2,$ distributivity of $\circ$
4	$\circ\left(\bigvee_{wR^*w'} \bigvee_{w'Rw''} \widehat{w''} \Rightarrow \bigvee_{wR^*w'} \widehat{w'}\right)$	$N_\circ, R^* \circ R \subseteq R^*$
5	$\circ\left(\bigvee_{wR^*w'} \bigvee_{w'Rw''} \widehat{w''}\right) \Rightarrow \circ\left(\bigvee_{wR^*w'} \widehat{w'}\right)$	$4, \mathbf{K}_\circ$
6	$\bigvee_{wR^*w'} \widehat{w'} \Rightarrow \circ\left(\bigvee_{wR^*w'} \widehat{w'}\right)$	$3, 5$
7	$\Box\left(\bigvee_{wR^*w'} \widehat{w'} \Rightarrow \circ\left(\bigvee_{wR^*w'} \widehat{w'}\right)\right)$	$6, N_\Box$
8	$\bigvee_{wR^*w'} \widehat{w'} \Rightarrow \Box\left(\bigvee_{wR^*w'} \widehat{w'}\right)$	$7, A4$
9	$\widehat{w} \Rightarrow \Box\left(\bigvee_{wR^*w'} \widehat{w'}\right)$	$8, Id_W \subseteq R^*$

Conclusion of the completeness proof

$M = \langle W, R, \{w_0\}, V\rangle$, $\varphi \in w_0$,

We want an $s = w_0 w_1 \ldots w_n \ldots$ s.t. $(\psi \mathsf{U} \chi) \in w_i$ entails $\chi \in w_{i+j}$ for some j.

Let $\{(\psi \mathsf{U} \chi) : (\psi \mathsf{U} \chi) \in \mathrm{Cl}(\varphi)\} = \{(\psi_0 \mathsf{U} \chi_0), \ldots, (\psi_{n-1} \mathsf{U} \chi_{n-1})\}$.

Step 0: We fix s to start at w_0: $s_0 = w_0$

Step $i+1$: Let $s_i = w_0 \ldots w_k$. Let $j = i \bmod n$.

- If $(\psi_j \mathsf{U} \chi_j) \in w_k$ and $\chi_j \notin w_k$,

  we choose $w' \in R^*(w_k)$ s.t. $\chi_j \in w'$ and put $s_{i+1} = s_i w_{k+1} \ldots w'$
- Otherwise, $s_{i+1} = s_i w'$ with an arbitrary $w' \in R(w_k)$.

Proposition 9 (Truth Lemma) $\sigma_s, i \models \psi$ iff $\psi \in s_i$ for all $\psi \in \mathrm{Cl}(\varphi)$, $i < \omega$.

The End