Classification of $PLTL$-definable properties and their canonical forms

$PLTL$ models and ω-languages

Linear models $\sigma : \omega \to \mathcal{P}(\mathbf{L})$ are ω-words in Σ^ω, where $\Sigma = \mathcal{P}(\mathbf{L})$.

A property $=$ a set of behaviours of a system $=$ an ω-language.

Definition 1 A property L is definable in $PLTL$ if there is a formula φ s.t.

$$L = \{\sigma : \sigma, 0 \models \varphi\}.$$

Notation

$\alpha \in \Sigma^* \cup \Sigma^+$, $\beta \in \Sigma^*$

$\beta \prec \alpha \leftrightarrow (\exists \gamma \in \Sigma^\omega \cup \Sigma^+)(\beta \cdot \gamma = \alpha)$ — β is a (proper) prefix of α

$\mathsf{pref}(\alpha) = \{\beta \in \Sigma^* : \beta \prec \alpha\}$

$L \subseteq \Sigma^\omega$ or $L \subseteq \Sigma^*$

$$\mathsf{pref}(L) = \bigcup_{\alpha \in L} \mathsf{pref}(\alpha)$$

$L \subseteq \Sigma^*$

$\mathsf{A}(L) = \{\alpha \in \Sigma^\omega : \mathsf{pref}(\alpha) \subseteq L\}$

$\mathsf{E}(L) = \{\alpha \in \Sigma^\omega : \mathsf{pref}(\alpha) \cap L \neq \emptyset\}$

More notation

$L \subseteq \Sigma^*$

$$\mathsf{A}_f(L) = \{\alpha \in \Sigma^* : \mathsf{pref}(\alpha) \subseteq L\}$$

$$\mathsf{E}_f(L) = \{\alpha \in \Sigma^* : \mathsf{pref}(\alpha) \cap L \neq \emptyset\}$$

$$\mathsf{P}(L) = \{\alpha \in \Sigma^\omega : \mathsf{pref}(\alpha) \setminus L \text{ is finite}\}$$

$$\mathsf{R}(L) = \{\alpha \in \Sigma^\omega : \mathsf{pref}(\alpha) \cap L \text{ is infinite}\}$$

Let $\overline{L} = \Sigma^\omega \setminus L$, resp. $\Sigma^* \setminus L$, for $L \subseteq \Sigma^\omega$, resp. $L \subseteq \Sigma^*$.

Exercise 1 Prove that $\mathsf{E}(L) = \overline{\mathsf{A}(\overline{L})}$, $\mathsf{E}_f(L) = \overline{\mathsf{A}_f(\overline{L})}$ and $\mathsf{P}(L) = \overline{\mathsf{R}(\overline{L})}$ for all $L \subseteq \Sigma^*$.

Exercise 2 (monotonicity of A, E, A_f, E_f, R and P) Prove that $L \subseteq M \subseteq \Sigma^*$ entails $\mathsf{X}(L) \subseteq \mathsf{X}(M)$ for $\mathsf{X} \in \{\mathsf{A}, \mathsf{E}, \mathsf{A}_f, \mathsf{E}_f, \mathsf{R}, \mathsf{P}\}$.

Definition of the primitive classes of properties

$L \subseteq \Sigma^\omega$ is a

safety	property, if	$L = \mathsf{A}(M)$	for some $M \subseteq \Sigma^*$
guarantee	-"-	$L = \mathsf{E}(M)$	-"-
persistence	-"-	$L = \mathsf{P}(M)$	-"-
recurrence	-"-	$L = \mathsf{R}(M)$	-"-

On safety properties

$\alpha \in \mathsf{A}(L)$ means that at no finite step i we observe $a_0 \ldots a_i \in \mathsf{pref}(\alpha) \setminus L$

L - the set of "good" histories;

α is "safe", if all the histories are good, i.e., nothing "bad" happens.

If $\pi \in \mathbf{L}$ is a past formula and $\sigma_h \in \mathcal{P}(\mathbf{L})^n$, $\sigma_t \in \mathcal{P}(\mathbf{L})^\omega$, then

$\sigma_h \cdot \sigma_t, |\sigma_h| - 1 \models \pi$ depends only on σ_h.

Definition 2 $\sigma_h \models \pi$ stands for $\exists \sigma_t \in \mathcal{P}(\mathbf{L})^\omega$ such that $\sigma_h \cdot \sigma_t, |\sigma_h| - 1 \models \pi$.

Let L_π denote $\{\sigma \in \mathcal{P}(\mathbf{L})^* : \sigma \models \pi\}$.

Then $\Box\pi$ defines the safety property $\mathsf{A}(L_\pi)$.

The vast majority of practically relevant properties are safety properties

Liveness is informally regarded as the complement of safety.

Definition 3 $L \subseteq \Sigma^\omega$ is a liveness property, if for every $\sigma \in \Sigma^*$ there exists a $\sigma' \in \Sigma^\omega$ s.t. $\sigma \cdot \sigma' \in L$, that is

Every finite σ can be extended to a behaviour which has the property L.

Exercise 3 Prove that if L is both a safety and a liveness property, then $L = \Sigma^\omega$.

Examlpe 1 $\Box(p \Rightarrow \Diamond q)$ - "every q is followed by a p" - is a liveness property.

A bound on q: "every q is followed by a p within k steps": $\Box(p \Rightarrow \bigvee_{l \leq k} \circ^l q)$

Exercise 4 This property is indeed safety. Write it in the form $\Box\pi$ with a past π.

Back to the primitive classes of properties

$L \subseteq \Sigma^\omega$ is a

safety	property, if	$L = \mathsf{A}(M)$	for some $M \subseteq \Sigma^*$
guarantee	-"-	$L = \mathsf{E}(M)$	-"-
persistence	-"-	$L = \mathsf{P}(M)$	-"-
recurrence	-"-	$L = \mathsf{R}(M)$	-"-

A characterization of safety/guarantee properties

Proposition 1 $L = \mathsf{A}(\mathsf{pref}(L))$ for safety properties L.

Proof: Let $L = \mathsf{A}(M)$. Then $\mathsf{pref}(L) \subseteq M$ and $\mathsf{A}(\mathsf{pref}(L)) \subseteq \mathsf{A}(M) = L$. To prove $L \subseteq \mathsf{A}(\mathsf{pref}(L))$, note that $\alpha \in L$ implies $\mathsf{pref}(\alpha) \subseteq \mathsf{pref}(L)$ by monotonicity and, consequently $\alpha \in \mathsf{A}(\mathsf{pref}(L))$. $\dashv$

Corollary 1 $L = \mathsf{E}\left(\overline{\mathsf{pref}(\overline{L})}\right)$ for guarantee properties L.

Closedness under $\cup$ and $\cap$ of the safety and guarantee classes

Obviously $\mathsf{A}(L_1) \cap \mathsf{A}(L_2) = \mathsf{A}(L_1 \cap L_2)$ for all $L_1, L_2 \subseteq \Sigma^*$.

Proposition 2 $\mathsf{A}(L_1) \cup \mathsf{A}(L_2) = \mathsf{A}(\mathsf{A}_f(L_1) \cup \mathsf{A}_f(L_2))$.

Proof: $\subseteq$: Let $i \in \{1, 2\}$, $\alpha \in \mathsf{A}(L_i)$. Then $\beta \in \mathsf{pref}(\alpha)$ implies $\beta \in \mathsf{A}_f(L_i)$, whence $\mathsf{pref}(\alpha) \subseteq \mathsf{A}_f(L_i)$. Then $\alpha \in \mathsf{A}(\mathsf{A}_f(L_i)) \subseteq \mathsf{A}(\mathsf{A}_f(L_1) \cup \mathsf{A}_f(L_2))$.

$\supseteq$: Let $\alpha \in \mathsf{A}(\mathsf{A}_f(L_1) \cup \mathsf{A}_f(L_2))$. Then $\mathsf{pref}(\alpha) \subseteq \mathsf{A}_f(L_1) \cup \mathsf{A}_f(L_2)$.

Since $\mathsf{pref}(\alpha)$ is infinite, either $\mathsf{pref}(\alpha) \cap \mathsf{A}_f(L_1)$ or $\mathsf{pref}(\alpha) \cap \mathsf{A}_f(L_2)$ is infinite.

Let $\mathsf{pref}(\alpha) \cap \mathsf{A}_f(L_i)$ be infinite. Then $\mathsf{pref}(\alpha) \subseteq \mathsf{A}_f(L_i)$.

This implies $\mathsf{pref}(\alpha) \subseteq L_i$, whence $\alpha \in \mathsf{A}(L_i)$.

Finally $\mathsf{A}(L_1) \cup \mathsf{A}(L_2) \supseteq \mathsf{A}(\mathsf{A}_f(L_1) \cup \mathsf{A}_f(L_2))$. $\dashv$

Closedness of under $\cup$ and $\cap$ of the recurrence and persistence classes

Obviously

$$\mathsf{R}(L) \cup \mathsf{R}(M) = \mathsf{R}(L \cup M) \text{ and } \mathsf{P}(L) \cap \mathsf{P}(M) = \mathsf{P}(L \cap M)$$

for all $L, M \subseteq \Sigma^*$.

Definition 4

$$\mathrm{ex}(\alpha, L) = \{\beta \in L : \alpha \prec \beta\}$$

$\mathsf{minex}(\alpha, L)$ is the set of the shortest words in $\mathrm{ex}(\alpha, L)$

$$\mathsf{minex}(M, L) = \bigcup_{\alpha \in M} \mathsf{minex}(\alpha, L)$$

Proposition 3 $\mathsf{R}(M) \cap \mathsf{R}(L) = \mathsf{R}(\mathsf{minex}(M, L))$ for all $M, L \subseteq \Sigma^*$.

Corollary 2 $\mathsf{P}(M) \cup \mathsf{P}(L) = \mathsf{P}\left(\overline{\mathsf{minex}(\overline{M}, \overline{L})}\right)$ for all $M, L \subseteq \Sigma^*$.

$\mathsf{R}(M) \cap \mathsf{R}(L) = \mathsf{R}(\mathsf{minex}(M, L))$: **Proof**

$\supseteq$: Let $\alpha \in \mathsf{R}(\mathsf{minex}(M, L))$, i.e., let $\mathsf{pref}(\alpha) \cap \mathsf{minex}(M, L)$ be infinite.

Since $\mathsf{minex}(M, L) \subseteq L$, $\mathsf{pref}(\alpha) \cap L$ is infinite too, whence $\alpha \in \mathsf{R}(L)$.

$\beta_1, \beta_2 \in \mathsf{pref}(\alpha) \cap \mathsf{minex}(M, L)$ implies $\beta_1 \preceq \beta_2$ or $\beta_2 \preceq \beta_1$.

Therefore, different $\beta \in \mathsf{pref}(\alpha) \cap \mathsf{minex}(M, L)$ are the shortest extensions of different $\gamma \in M$.

Hence, since $\mathsf{pref}(\alpha) \cap \mathsf{minex}(M, L)$ is infinite, $\mathsf{pref}(\alpha) \cap M$ is infinite too, i.e., $\alpha \in \mathsf{R}(M)$.

$\subseteq$: Let $\alpha \in \mathsf{R}(M) \cap \mathsf{R}(L)$. Then $\mathsf{pref}(\alpha) \cap M$ and $\mathsf{pref}(\alpha) \cap L$ are infinite.

Choose an arbitrary $n < \omega$.

There exist $\beta \in \mathsf{pref}(\alpha) \cap M$ and $\gamma \in \mathsf{pref}(\alpha) \cap L$ s.t. $n < |\beta|$, and $\beta \prec \gamma$.

Given such β and γ, $\mathrm{ex}(\beta, L) \neq \emptyset$ and $\beta \prec \delta \preceq \gamma$ for some $\delta \in \mathsf{minex}(\beta, L)$.

Furthermore $\delta \in \mathsf{pref}(\alpha) \cap \mathsf{minex}(M, L)$ and $|\delta| > n$.

Hence $\mathsf{pref}(\alpha) \cap \mathsf{minex}(M, L)$ is infinite, i.e. $\alpha \in \mathsf{R}(\mathsf{minex}(M, L))$.

Inclusions between the classes

Exercise 5 Prove that $\mathsf{E}(L) = \mathsf{R}(\mathsf{E}_f(L))$ and $\mathsf{A}(L) = \mathsf{P}(\mathsf{A}_f(L))$ for all $L \subseteq \Sigma^*$.

Proposition 4 $\mathsf{A}(L) = \mathsf{R}(\mathsf{A}_f(L))$ for all $L \subseteq \Sigma^*$.

Proof: $\supseteq$: Let $\alpha \in \mathsf{R}(\mathsf{A}_f(L))$. Then $\mathsf{pref}(\alpha) \cap \mathsf{A}_f(L)$ is infinite.

Choose an arbitrary $\beta \in \mathsf{pref}(\alpha)$.

Then there is a $\gamma \in \mathsf{pref}(\alpha) \cap \mathsf{A}_f(L)$ s.t. $\beta \prec \gamma$, which implies $\beta \in L$. Hence $\mathsf{pref}(\alpha) \subseteq L$, i.e., $\alpha \in \mathsf{A}(L)$.

$\subseteq$: Let $\alpha \in \mathsf{A}(L)$, that is, $\mathsf{pref}(\alpha) \subseteq L$. Then $\mathsf{pref}(\alpha) \subseteq \mathsf{A}_f(L)$.

Since $\mathsf{pref}(\alpha)$ is infinite, this entails $\alpha \in \mathsf{R}(\mathsf{A}_f(L))$. $\dashv$

Corollary 3 $\mathsf{E}(L) = \mathsf{P}(\mathsf{E}_f(L))$ for all $L \subseteq \Sigma^*$.

Summary

Complementation between the classes

The complement of a safety property is a guarantee property and vice versa.

The complement of a recurrence property is a persistence property and vice versa.

Closedness under $\cup$ and $\cap$

The classes of safety, guarantee, persistence and recurrence properties are all closed under $\cup$ and $\cap$.

Inclusion of the classes

A safety property is both a recurrence and a persistence property as well.

A guarantee property is similarly both a recurrence and a persistence property.

The compound classes

Definition 5 L is an obligation property, if L is a combination of safety and guarantee properties by $\cup$ and $\cap$.

Proposition 5 Every obligation property has the form $\bigcap_i \mathsf{A}(L_i) \cup \mathsf{E}(M_i)$ for some $L_i, M_i \subseteq \Sigma^*$.

Corollary 4 Every obligation property is both a recurrence and a persistence property.

Definition 6 L is a reactivity property, if L is a combination of recurrence and persisitence properties by $\cup$ and $\cap$.

Proposition 6 Every reactivity property has the form $\bigcap_i \mathsf{R}(L_i) \cup \mathsf{P}(M_i)$ for some $L_i, M_i \subseteq \Sigma^*$.

The safety-liveness classification

Definition 7 Recall that $L \subseteq \Sigma^\omega$ is a liveness property, if $\mathsf{pref}(L) = \Sigma^*$.

Proposition 7 Every $X \subseteq \Sigma^\omega$ has the form $S \cap L$ for some safety property S and some liveness property L.

Proof: We put $S = \mathsf{A}(\mathsf{pref}(X))$ and $L = X \cup \mathsf{E}\left(\overline{\mathsf{pref}(X)}\right)$.

Let $\beta \in \Sigma^*$. If $\beta \in \mathsf{pref}(X)$, then β has an infinite extension in X. Otherwise, all the infinite extensions of β are in $\mathsf{E}(\overline{\mathsf{pref}(X)})$. Hence L is a liveness property.

Obviously $S \cap \mathsf{E}(\overline{\mathsf{pref}(X)}) = \emptyset$. Hence $S \cap L = S \cap X$. Now $S \cap L = X$ follows from $X \subseteq S = \mathsf{A}(\mathsf{pref}(X))$, which is established by a direct check. $\dashv$

Definition 8 $\mathsf{A}(\mathsf{pref}(X))$ is called the safety closure of X. $\mathsf{E}(\overline{\mathsf{pref}(X)})$ is called the liveness extension of X.

Back to $PLTL$

Until now nothing depended on the expressibility of properties in $PLTL$

Let $\Sigma = \mathcal{P}(\mathbf{L})$.

Recall that $L_\pi = \{\sigma \in \Sigma^* : \sigma \models \pi\}$ for past π. Then

$$\mathsf{A}_f(L_\pi) = L_{\boxminus\pi} \text{ and } \mathsf{E}_f(L_\pi) = L_{\diamondminus\pi}.$$

(A_f and E_f are about proper prefixes; $\diamondminus$ and $\boxminus$ have the strict interpretation.)

Let

$$L_\varphi = \{\sigma \in \Sigma^\omega : \sigma, 0 \models \varphi\}$$

for φ with future temporal operators. Then

$$\mathsf{A}(L_\pi) = L_{\Box\pi},\ \mathsf{E}(L_\pi) = L_{\Diamond\pi},\ \mathsf{R}(L_\pi) = L_{\Box\Diamond\pi} \text{ and } \mathsf{P}(L_\pi) = L_{\Diamond\Box\pi}$$

Proof: Exercise. $\dashv$

Complementation and closedness under $\cap$ and $\cup$ in terms of $PLTL$

$$\mathsf{A}_f(L_\pi) = L_{\boxminus\pi} \text{ and } \mathsf{E}_f(L_\pi) = L_{\diamondminus\pi}$$

$$\mathsf{A}(L_\pi) = L_{\Box\pi},\ \mathsf{E}(L_\pi) = L_{\Diamond\pi},\ \mathsf{R}(L_\pi) = L_{\Box\Diamond\pi} \text{ and } \mathsf{P}(L_\pi) = L_{\Diamond\Box\pi}$$

Complementation:

$\overline{\mathsf{A}(L_\pi)} = \mathsf{E}(\overline{L_\pi})$, $\overline{\mathsf{P}(L_\pi)} = \mathsf{R}(\overline{L_\pi})$ $\neg\Box\pi \Leftrightarrow \Diamond\neg\pi,\ \neg\Diamond\Box\pi \Leftrightarrow \Box\Diamond\neg\pi$

Closedness under $\cap$ and $\cup$:

for safety properties

$\mathsf{A}(L_{\pi_1}) \cap \mathsf{A}(L_{\pi_2}) = \mathsf{A}(L_{\pi_1} \cap L_{\pi_2})$ $\Box\pi_1 \wedge \Box\pi_2 \Leftrightarrow \Box(\pi_1 \wedge \pi_2)$

$\mathsf{A}(L_{\pi_1}) \cup \mathsf{A}(L_{\pi_2}) = \mathsf{A}(\mathsf{A}_f(L_{\pi_1}) \cup \mathsf{A}_f(L_{\pi_2}))$ $\Box\pi_1 \vee \Box\pi_2 \Leftrightarrow \Box(\boxminus\pi_1 \vee \boxminus\pi_2)$

for guarantee properties

$\mathsf{E}(L_{\pi_1}) \cup \mathsf{E}(L_{\pi_2}) = \mathsf{E}(L_{\pi_1} \cup L_{\pi_2})$ $\Diamond\pi_1 \vee \Diamond\pi_2 \Leftrightarrow \Diamond(\pi_1 \vee \pi_2)$

$\mathsf{E}(L_{\pi_1}) \cap \mathsf{E}(L_{\pi_2}) = \mathsf{E}(\mathsf{E}_f(L_{\pi_1}) \cap \mathsf{E}_f(L_{\pi_2}))$ $\Diamond\pi_1 \wedge \Diamond\pi_2 \Leftrightarrow \Diamond(\diamondminus\pi_1 \wedge \diamondminus\pi_2)$

Closedness under $\cap$ and $\cup$ for recurrence and persistence

Proposition 8 $\mathsf{minex}(L_{\pi_1}, L_{\pi_2}) = \{\sigma \in \Sigma^* : \sigma \models \pi_2 \wedge (\neg\pi_2 \mathsf{S} \pi_1)\}$

Closedness under $\cap$ and $\cup$:

for recurrence properties

$\mathsf{R}(L_{\pi_1}) \cup \mathsf{R}(L_{\pi_2}) = \mathsf{R}(L_{\pi_1} \cup L_{\pi_2})$ $\qquad \Box\Diamond\pi_1 \vee \Box\Diamond\pi_2 \Leftrightarrow \Box\Diamond(\pi_1 \vee \pi_2)$

$\mathsf{R}(L_{\pi_1}) \cap \mathsf{R}(L_{\pi_2}) = \mathsf{R}(\mathsf{minex}(L_{\pi_1}, L_{\pi_2}))$ $\qquad \Box\Diamond\pi_1 \wedge \Box\Diamond\pi_2 \Leftrightarrow$

$\Box\Diamond(\pi_1 \wedge (\neg\pi_1 \mathsf{S} \pi_2))$

for persistence properties

$\mathsf{P}(L_{\pi_1}) \cap \mathsf{P}(L_{\pi_2}) = \mathsf{P}(L_{\pi_1} \cap L_{\pi_2})$ $\qquad \Diamond\Box\pi_1 \wedge \Diamond\Box\pi_2 \Leftrightarrow \Diamond\Box(\pi_1 \wedge \pi_2)$

$\mathsf{P}(L_{\pi_1}) \cup \mathsf{P}(L_{\pi_2}) = \mathsf{P}\left(\overline{\mathsf{minex}(\overline{L_{\pi_1}}, \overline{L_{\pi_2}})}\right)$ $\qquad \Diamond\Box\pi_1 \vee \Diamond\Box\pi_2 \Leftrightarrow$

$\Diamond\Box(\pi_1 \vee \neg(\pi_1 \mathsf{S} \neg\pi_2))$

Inclusions of the classes in terms of $PLTL$

$\mathsf{A}(L_\pi) = \mathsf{P}(\mathsf{A}_f(L_\pi)) = \mathsf{R}(\mathsf{A}_f(L_\pi)) \quad \Box\pi \Leftrightarrow \Diamond\Box\boxminus\pi,\ \Box\pi \Leftrightarrow \Box\Diamond\boxminus\pi$

$\mathsf{E}(L_\pi) = \mathsf{P}(\mathsf{E}_f(L_\pi)) = \mathsf{R}(\mathsf{E}_f(L_\pi)) \quad \Diamond\pi \Leftrightarrow \Diamond\Box\diamondminus\pi,\ \Diamond\pi \Leftrightarrow \Box\Diamond\diamondminus\pi$

Canonical forms for $PLTL$-definable properties: overview

So far we know that if π is past, then

A$(L_\pi) = L_{\Box\pi}$, and therefore $\Box\pi$ defines a safety property

P$(L_\pi) = L_{\Diamond\Box\pi}$, and therefore $\Box\Diamond\pi$ defines a persistence property, etc.

It can be shown that, regardless of the syntax of φ,

if φ defines a safety property, then $0 \models \varphi \Leftrightarrow \Box\pi$ for some past π

if φ defines a persistence property, then $0 \models \varphi \Leftrightarrow \Diamond\Box\pi$ for some past π, etc.

This was first done using ω-automata which accept regular ω-languages.

Regular ω-languages

Definition 9 An ω-language $L \subseteq \Sigma^\omega$ is regular, if it has the form

$$\bigcup_i M_i \cdot L_i^\omega$$

for some regular $M_i, L_i \subseteq \Sigma^*$.

Proposition 9 All $PLTL$ definable properties are regular.

Proposition 10 A property $L \subseteq \Sigma^\omega$ is regular iff it is accepted by an ω-automaton.

ω-Automata

$A = \langle Q, \Sigma, \delta, q_0, Acc \rangle$

$Q \neq \emptyset$ is a finite set of states, $\quad q_0 \in Q$ is the initial state

Σ is a finite alphabet ($= \mathcal{P}(\mathbf{L})$ in our case)

$\delta : Q \times \Sigma \to \mathcal{P}(Q) \setminus \{\emptyset\}$ is a transition function

Acc is an acceptance condition

$$\mathrm{run}_A(\sigma) = \{r \in Q^\omega : r_0 = q_0 \text{ and } r_{i+1} \in \delta(r_i, \sigma_i) \text{ for all } i \in \omega\}$$

The standard extension of δ to a function of type $Q \times \Sigma^* \to \mathcal{P}(Q) \setminus \{\emptyset\}$:

$\delta(q, \sigma)$ is the set of the states that are reachable from q upon reading σ.

$$\mathrm{inf}(r) = \{q : r_i = q \text{ for infinitely many } i \in \omega\}$$

Streett automata: $Acc \subseteq \mathcal{P}(Q) \times \mathcal{P}(Q)$; Word $\sigma \in \Sigma^\omega$ is accepted, if

$$(\exists r \in \mathrm{run}_A(\sigma))(\forall \langle X, Y \rangle \in Acc)(\mathrm{inf}(r) \cap X \neq \emptyset \to \mathrm{inf}(r) \cap Y \neq \emptyset).$$

Types of automata, depending on Acc

...-automaton	Acc	condition for accepting $\sigma \in \Sigma^\omega, \Sigma^*$
Mealy	$F \subseteq Q$	$\delta(s_0, \sigma) \cap F \neq \emptyset$
Büchi	$F \subseteq Q$	$(\exists r \in \mathrm{run}_A(\sigma))\, \mathrm{inf}(r) \cap F \neq \emptyset$
generalised Büchi	$\mathcal{F} \subseteq \mathcal{P}(Q)$	$(\exists r \in \mathrm{run}_A(\sigma))(\forall F \in \mathcal{F})\, \mathrm{inf}(r) \cap F \neq \emptyset$
Müller	$Acc \subseteq \mathcal{P}(Q)$	$(\exists r \in \mathrm{run}_A(\sigma))\, \mathrm{inf}(r) \in Acc$
Streett	$Acc \subseteq \mathcal{P}(Q)^2$	$(\exists r \in \mathrm{run}_A(\sigma))(\forall \langle X, Y\rangle \in Acc)$ $(\mathrm{inf}(r) \cap X \neq \emptyset \rightarrow \mathrm{inf}(r) \cap Y \neq \emptyset)$
parity	$c : Q \rightarrow \{1, \ldots, n\}$	$(\exists r \in \mathrm{run}_A(\sigma))\, \min_{q \in \mathrm{inf}(r)} c(q)$ is even.

Reference: Wolfgang Thomas. Automata on infinite objects. In: *Handbook of Theoretical Computer Science, volume B*, pp 133-192. Elsevier, 1990.

Canonical forms for regular properties

Theorem 1

If $L \subseteq \Sigma^\omega$ is a regular safety property, then there exists a regular $M \subseteq \Sigma^*$ such that $L = \mathsf{A}(M)$.

Similarly, if L is a regular recurrence property, then $L = \mathsf{P}(M)$ for some regular M.

Every regular property has the form

$$\bigcap_i \mathsf{R}(M_i) \cup \mathsf{P}(N_i)$$

for some regular $M_i, N_i \subseteq \Sigma^*$.

Sketch of the proof

Let automaton $A = \langle Q, \Sigma, \delta, q_0, Acc\rangle$ accept L. Let

$$M_q = \{\sigma \in \Sigma^* : \delta(q_0, \sigma) = q\} \text{ for every } q \in Q.$$

For safety L, $L = \mathsf{A}(M)$, where

$$M = \bigcup \{M_q : q \text{ occurs in an accepting run for some } \sigma \in L\}.$$

For a recurrence L, A can be chosen so that $X = Q$ for all $\langle X, Y\rangle \in Acc$.

$$\text{Then } L = \bigcap_{\langle Q,Y\rangle \in Acc} \mathsf{R}(\cup_{q\in Y} M_q).$$

For a reactivity property L we have

$$L = \bigcap_{\langle X,Y\rangle \in Acc} \mathsf{P}(\overline{\cup_{q\in X} M_q}) \cup \mathsf{R}(\cup_{q\in Y} M_q).$$

Reference: Zohar Manna, Amir Pnueli, The anchored version of the temporal framework. In: LNCS 354, pp. 201-284, 1989.

Canonical forms for $PLTL$-definable properties

Theorem 2

If $L \subseteq \mathcal{P}(\mathbf{L})^{\omega}$ is a $PLTL$-definable safety property, then there exists a past formula $\pi \in \mathbf{L}$ such that $L = \mathsf{A}(L_\pi)$, that is, L is defined by $\Box\pi$.

Similarly, if L is a recurrence property, then there exists a past π such that $L = \mathsf{R}(L_\pi)$, that is, L is defined by $\Box\Diamond\pi$.

Every $PLTL$-definable property is definable by a formula of the form

$$\bigwedge_i \Diamond\Box\pi_i \Rightarrow \Diamond\Box\pi'_i$$

where π_i, π'_i are past formulas.

Proofs by means of the separation theorem: safety

Let $L = L_\varphi$.

$$0 \models \varphi \Leftrightarrow \Box \diamondminus(\mathsf{I} \wedge \varphi) \text{ and } 0 \models \varphi \Leftrightarrow \Diamond \diamondminus(\mathsf{I} \wedge \varphi).$$

Let $\bigvee\limits_i \pi_i \wedge \circ\varphi_i$ be a separated equivalent to $\diamondminus(\mathsf{I} \wedge \varphi)$.

We can assume all the φ_is to be satisfiable.

$$\models_{PLTL} \Box \left(\bigvee_i \pi_i \wedge \circ\varphi_i \right) \Rightarrow \Box \bigvee_i \pi_i, \text{ which implies } 0 \models \varphi \Rightarrow \Box \bigvee_i \pi_i.$$

Using that φ defines a safety property, we prove that

$$0 \models \Box \left(\bigvee_i \pi_i \right) \Rightarrow \varphi.$$

Let $\sigma, 0 \models \Box \bigvee_i \pi_i$. Then for every $k < \omega$ there is an $i < \omega$ s.t. $\sigma_0 \ldots \sigma_k \models \pi_i$. Let $\sigma' \in \mathcal{P}(\mathbf{L})^\omega$ and let $\sigma', 0 \models \varphi_i$. Then $\sigma_0 \ldots \sigma_k \cdot \sigma'$ is an infinite extension of $\sigma_0 \ldots \sigma_k$ and $\sigma_0 \ldots \sigma_k \cdot \sigma', k \models \pi_i \wedge \circ\varphi_i$, which implies that

$$\sigma_0 \ldots \sigma_k \cdot \sigma', 0 \models \varphi \text{ because } 0 \models \varphi \Leftrightarrow \Diamond \left(\bigvee_i \pi_i \wedge \circ\varphi_i \right).$$

Hence every prefix $\sigma_0 \ldots \sigma_k$ of a σ that satisfies $\bigvee_i \pi_i$ has an infinite extension which satisfies φ. Since φ defines a safety property, $\sigma, 0 \models \varphi$.

Hence $0 \models \Box \left(\bigvee_i \pi_i \right) \Rightarrow \varphi$.

Proofs by means of the separation theorem: recurrence and reactivity

There is no syntactical proof for recurrence that I know.

There is a syntactical proof for reactivity, based on separation. (Guelev, *Journal of Logic and Computation*, 2008.)

There is an earlier proof for reactivity, by Mark Reynolds, LICS 2000, which is a mix of semantic transformations and application of another variant of separation, which applies to Dedekind-complete time models.

A canonical form for $PLTL$-definable liveness properties

Theorem 3 A $PLTL$-definable property is a liveness property iff it is definable by a formula of the form $\Diamond\left(\bigvee_i \pi_i \wedge \circ\varphi_i\right)$ in which φ_i are satisfiable future formulas, π_i are past formulas, and $\bigvee_i \pi_i$ is valid.

Proof: $\leftarrow$ - Direct check. $\rightarrow$ Let φ define the considered liveness property and $\psi = \bigvee_i \pi_i \wedge \circ\varphi_i$ be a separated equivalent to $\Diamond\!\!\!-(\mathsf{I} \wedge \varphi)$. Then for all $\sigma \in \mathcal{P}(\mathbf{L})^\omega$ we have both

$$\sigma, 0 \models \varphi \Leftrightarrow \Diamond\psi \text{ and } \sigma, 0 \models \varphi \Leftrightarrow \Box\psi.$$

Let $\sigma \in \mathcal{P}(\mathbf{L})^*$. Since φ is a liveness property, there exists a $\gamma \in \mathcal{P}(\mathbf{L})^\omega$ s. t.

$$\sigma \cdot \gamma, 0 \models \Box\left(\bigvee_i \pi_i \wedge \circ\varphi_i\right),$$

which entails that $\sigma \models \bigvee_i \pi_i$. $\dashv$

The End