From Small Space to Small Width in Resolution

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Joint work with Yuval Filmus, Massimo Lauria, Jakob Nordström, and Marc Vinyals
Resolution

- **Input:** CNF formula $F$

  
  \[(x \lor \overline{y} \lor z) \land (\overline{y} \lor \overline{z}) \land (x \lor y) \land (\overline{x} \lor \overline{z}) \land (\overline{x} \lor z)\]

- **Resolution rule:**

  \[
  \begin{array}{c}
  C \lor x \\
  D \lor \overline{x} \\
  \hline
  C \lor D
  \end{array}
  \]

- **Goal:** Proof of unsatisfiability (refutation) $= \text{Derive empty clause } \bot$

Refer to clauses of formula as **axioms**
Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
- DAG

\[
\begin{align*}
1. & \quad x \lor \overline{y} \lor z & \text{Axiom} \\
2. & \quad \overline{y} \lor \overline{z} & \text{Axiom} \\
3. & \quad x \lor \overline{y} & \text{Res}(1, 2) \\
4. & \quad x \lor y & \text{Axiom} \\
5. & \quad x & \text{Res}(3, 4) \\
6. & \quad \overline{x} \lor \overline{z} & \text{Axiom} \\
7. & \quad \overline{x} \lor z & \text{Axiom} \\
8. & \quad \overline{x} & \text{Res}(6, 7) \\
9. & \quad \bot & \text{Res}(5, 8)
\end{align*}
\]
Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
- DAG

Example:

\[
\begin{align*}
&x \lor \bar{y} \lor z \\
&\bar{y} \lor \bar{z} \\
&x \lor \bar{y} \\
&x \lor y \\
&x \\
&\bar{x} \lor \bar{z} \\
&\bar{x} \lor z \\
&\bar{x} \\
&\bot
\end{align*}
\]
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**
- Size
- Space
- Width
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step $t$: 
# clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$x \lor \overline{y} \lor z$</td>
<td>Axiom</td>
</tr>
<tr>
<td>2.</td>
<td>$\overline{y} \lor \overline{z}$</td>
<td>Axiom</td>
</tr>
<tr>
<td>3.</td>
<td>$x \lor \overline{y}$</td>
<td>Res(1, 2)</td>
</tr>
<tr>
<td>4.</td>
<td>$x \lor y$</td>
<td>Axiom</td>
</tr>
<tr>
<td>5.</td>
<td>$x$</td>
<td>Res(3, 4)</td>
</tr>
<tr>
<td>6.</td>
<td>$\overline{x} \lor \overline{z}$</td>
<td>Axiom</td>
</tr>
<tr>
<td>7.</td>
<td>$\overline{x} \lor z$</td>
<td>Axiom</td>
</tr>
<tr>
<td>8.</td>
<td>$\overline{x}$</td>
<td>Res(6, 7)</td>
</tr>
<tr>
<td>9.</td>
<td>$\perp$</td>
<td>Res(5, 8)</td>
</tr>
</tbody>
</table>

Size 9

Space

Width
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

Size: number of steps in refutation
Space: memory usage (at step $t$: 
# clauses before $t$ used after $t$)
Width: size of the largest clause

Example:
- Size 9
- Space
- Width
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

Size: number of steps in refutation
Space: memory usage (at step $t$: # clauses before $t$ used after $t$)
Width: size of the largest clause

Example:
- Size 9
- Space 0
- Width

Space at current step 0
Can represent refutation as
- annotated list or
- DAG

**Size**: number of steps in refutation

**Space**: memory usage (at step $t$: 
# clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:
- Size 9
- Space 1
- Width

Space at current step 1
Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step $t$:
# clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**
- Size 9
- Space 2
- Width

Space at current step 2
Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**
- Size 9
- Space 3
- Width

Space at current step 3
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size**: number of steps in refutation

**Space**: memory usage (at step $t$: 
# clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:
- Size 9
- Space 3
- Width

Space at current step 2
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

Size: number of steps in refutation
Space: memory usage (at step $t$: # clauses before $t$ used after $t$)
Width: size of the largest clause

Example:
- Size 9
- Space 3
- Width

Space at current step 3
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size**: number of steps in refutation

**Space**: memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:
- Size 9
- Space 3
- Width

Space at current step: 2
Resolution Size, Space, and Width

Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**
- Size 9
- Space 3
- Width

Space at current step 3
Resolution Size, Space, and Width

Can represent refutation as
  - annotated list or
  - DAG

**Size**: number of steps in refutation

**Space**: memory usage (at step $t$: 
  # clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:

Size  9
Space  4
Width

Space at current step  4

$x \lor \neg y \lor z$

$\neg y \lor \neg z$

$x \lor \neg y$

$x \lor y$

$x \lor \neg z$

$\bot$
Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step \( t \): # clauses before \( t \) used after \( t \))

**Width:** size of the largest clause

**Example:**

- Size 9
- Space 4
- Width

Space at current step 3
Can represent refutation as
- annotated list or
- DAG

**Size:** number of steps in refutation

**Space:** memory usage (at step $t$: 
# clauses before $t$ used after $t$)

**Width:** size of the largest clause

**Example:**
- Size 9
- Space 4
- Width

Space at current step 0
Resolution Size, Space, and Width

Can represent refutation as

- annotated list or
- DAG

**Size**: number of steps in refutation

**Space**: memory usage (at step \( t \):  
\# clauses before \( t \) used after \( t \))

**Width**: size of the largest clause

**Example**:

- Size \( 9 \)
- Space \( 4 \)
- Width \( 3 \)

Space at current step \( 0 \)

\[
x \lor \overline{y} \lor z
\]

\[
\overline{y} \lor \overline{z}
\]

\[
x \lor \overline{y}
\]

\[
x \lor y
\]

\[
x
\]

\[
x \lor \overline{z}
\]

\[
\overline{x} \lor \overline{z}
\]

\[
\overline{x} \lor z
\]

\[
\overline{x}
\]

\[
\bot
\]
Can represent refutation as
- annotated list or
- DAG

**Size**: number of steps in refutation

**Space**: memory usage (at step $t$: # clauses before $t$ used after $t$)

**Width**: size of the largest clause

**Example**:
- Size 9
- Space 4
- Width 3

Space at current step 0
Relation Between Width and Size/Space

**Width** helps us understand size and space
 Makes most sense for small width formulas — focus on $k$-CNFs

**Size:** Ben-Sasson and Wigderson ’99

$$\log(\text{Size}) \gtrsim \text{Width}$$

Proof by syntactically manipulating short refutation into narrow refutation

**Space:** Atserias and Dalmau ’03

$$\text{Space} \geq \text{Width}$$

More involved proof in terms of strategies for Ehrenfeucht-Fraïssé games
Relation Between Width and Size/Space

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More involved proof in terms of strategies for Ehrenfeucht-Fraïssé games

**Our result:** Simple purely syntactic proof

Razborov independently obtained a similar proof
Refutation presented on whiteboard
Refutation presented on whiteboard

- Write down axioms
Refutation presented on whiteboard

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Refutation presented on whiteboard

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Refutation presented on whiteboard

- Write down axioms
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
Whiteboard Interpretation of Space

Refutation presented on whiteboard
- Write down axioms
- Use resolution rule
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause
Refutation presented on whiteboard
- Write down axioms
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Refutation presented on whiteboard

- Write down axioms
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- Erase clause
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause

Logical expressions:
- $x \lor \neg y \lor z$
- $\neg y \lor \neg z$
- $x \lor \neg y$
- $x \lor \neg z$
- $\neg x \lor \neg z$
- $\neg x \lor z$
- $\neg x$
- $\bot$

$\bot$
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause
Refutation presented on whiteboard
- Write down axioms
- Use resolution rule
- Erase clause
Refutation presented on whiteboard

- Write down axioms
- Use resolution rule
- Erase clause
Whiteboard Interpretation of Space

Refutation presented on whiteboard
- Write down axioms
- Use resolution rule
- Erase clause

**Space:** max # clauses on board
No finite model theory and no Ehrenfeucht-Fraïssé games
Want to turn small-height whiteboard into small-width one

\[ x \lor y \lor z \lor v \lor w \]
\[ y \lor z \lor w \lor x \]
\[ x \lor y \lor \overline{y} \]
\[ x \lor y \lor \overline{z} \lor \overline{v} \lor \overline{w} \]

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Proof Complexity '14 6/14
Proof Idea in One Slide

No finite model theory and no Ehrenfeucht-Fraïssé games

Want to turn small-height whiteboard into small-width one

**Rotate** whiteboard and get narrow whiteboard

\[ x \lor \neg y \lor \neg z \lor v \lor w \]
\[ \neg y \lor \neg z \lor \neg w \lor x \]
\[ x \lor \neg y \]

\[ x \land \neg m \land \neg z \land \neg h \land x \]
\[ y \land \neg m \land \neg z \land \neg h \land y \]
In Somewhat More Detail...

\[ x \lor \overline{y} \lor z \\
\overline{y} \lor \overline{z} \]
In Somewhat More Detail...

\[ \neg((x \lor \overline{y} \lor z) \land (\overline{y} \lor z)) \]

- View clauses on whiteboard as CNF and negate

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In Somewhat More Detail...

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
In Somewhat More Detail...

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
- Expand the formula into CNF by distributing OR over ANDs
In Somewhat More Detail...

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
- Expand the formula into CNF by distributing OR over ANDs
- Remove trivial and redundant clauses
In Somewhat More Detail...

- View clauses on whiteboard as CNF and negate
- Apply DeMorgan’s rules
- Expand the formula into CNF by distributing OR over ANDs
- Remove trivial and redundant clauses
- Write CNF on whiteboard

Space (\# clauses) of Original \(\geq\) Width of Negated
Consequences of Negation

Negate every whiteboard and run refutation in reverse

Note: Empty whiteboard turns into contradiction and vice versa

Small space refutation is transformed into narrow one
Missing Details

Need two things

1. **Prove** we have backbone of resolution refutation
2. **Fill in** missing details (without blowing up width)

Proof by *case analysis* over derivation steps:

- Axiom download
- Resolution rule application
- Clause erasure
Clause Erasure

Original: Erasure weakens whiteboard

Right board weaker than left board

Original refutation

\[
\begin{align*}
\overline{y} \lor \overline{z} \\
x \lor \overline{y}
\end{align*}
\]
**Clause Erasure**

**Original:** Erasure weakens whiteboard

Right board weaker than left board

**Negated:** Negation inverts relation

Left board weaker than right board

Negated refutation run in reverse!
Can skip weaker whiteboards
Resolution Rule Application

Original: No change in semantic content

Original refutation

Original:

\( x \lor \overline{y} \lor z \)
\( \overline{y} \lor z \)

Negated:

\( x \lor \overline{y} \lor z \)
\( \overline{y} \lor z \)
\( x \lor \overline{y} \)

Note: No work done thus far!
Resolution Rule Application

Original refutation:

\[ x \lor \neg y \lor z \]
\[ \neg y \lor z \]

Negated refutation:

\[ x \lor \neg y \lor z \]
\[ \neg y \lor z \]
\[ x \lor \neg y \]

**Original:** No change in semantic content

**Negated:** No change in syntactic content (after prunning redundant clauses)

Whiteboard stays the same!
Resolution Rule Application

Original refutation

Original: No change in semantic content

Negated: No change in syntactic content (after pruning redundant clauses)

Whiteboard stays the same!

Negated refutation

Note: No work done thus far!
**Original**: Add axiom $A$ to whiteboard

\[
x \lor \overline{y} \lor z\\
\overline{y} \lor \overline{z}
\]

**Original refutation**
**Original:** Add axiom $A$ to whiteboard

**Negated:** For every literal $a \in A$ add $\overline{a}$ to all clauses of whiteboard

Use clauses $C \lor \overline{a}$ and $A$ to derive $C$
Original: Add axiom $A$ to whiteboard

Negated: For every literal $a \in A$ add $\overline{a}$ to all clauses of whiteboard

Use clauses $C \lor \overline{a}$ and $A$ to derive $C$

\[
\begin{align*}
\overline{y} \lor \overline{z} & \quad \overline{x} \lor y \\
\overline{x} \lor \overline{z} & \quad \overline{x} \lor z \\
& \quad \overline{x}
\end{align*}
\]

Adds constant width to derivation
**Original:** Add axiom $A$ to whiteboard

**Negated:** For every literal $a \in A$ add $\overline{a}$ to all clauses of whiteboard

Use clauses $C \lor \overline{a}$ and $A$ to derive $C$

$$
\begin{align*}
&\overline{y} \lor \overline{z} \\
&\overline{x} \lor y \\
&\overline{x} \lor z \\
&y
\end{align*}
$$

Adds constant width to derivation

**Theorem**

$$
\text{Space} \geq \text{Width}
$$

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**Open Problem: Similar Problem for Polynomial Calculus**

**Polynomial calculus**
Stronger proof system based on algebraic reasoning
Lines are polynomial equations instead of clauses
Degree of refutation analogous to width in resolution

**Size:** Impagliazzo, Pudlák, and Sgall ’99

\[ \log(\text{Size}) \gtrsim \text{Degree} \]
Polynomial calculus
Stronger proof system based on algebraic reasoning
Lines are polynomial equations instead of clauses
Degree of refutation analogous to width in resolution
Size: Impagliazzo, Pudlák, and Sgall ’99

\[ \log(\text{Size}) \preceq \text{Degree} \]

Open Problem
Is Space \( \geq \) Degree in polynomial calculus?

Original motivation for our work
We show our approach is unlikely to work (see paper for details)
Concluding Remarks

- Space upper bounds width in resolution [Atserias and Dalmau ’03]
- **This work:** New simple proof of this theorem
- **Open problem:** Space-degree relation in polynomial calculus?
Concluding Remarks

- Space upper bounds width in resolution [Atserias and Dalmau ’03]
- **This work:** New simple proof of this theorem
- **Open problem:** Space-degree relation in polynomial calculus?

Thank you for your attention!