Model Checking Lower Bounds for Simple Graphs

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Algorithmic Meta-Theorems

Positive results

- Problem X is tractable.

Negative results

- Problem X is hard.

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Main objective of today’s talk: barriers to meta-theorems:

“There exists a problem in class C that is hard”

Good news so far

- Most famous meta-theorem: Courcelle's theorem

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  Example:

  $$\exists S \forall x \forall y E(x,y) \to (x \in S \leftrightarrow y \notin S)$$

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  - Wider classes of problems?
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  Meta-theorems for clique-width, local treewidth,. . .

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  This can be extended to optimization versions of MSO.

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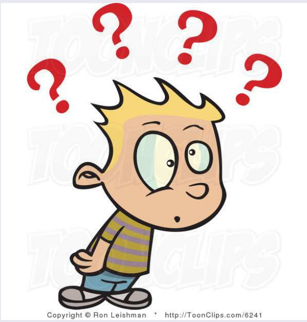

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This is the main question we are concerned with today.

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- Courcelle's theorem:

  There exists an algorithm which, given an MSO formula ϕ and a graph G with treewidth w decides if $G \models \phi$ in time $f(w, \phi)|G|$.

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- In fact, Frick and Grohe's lower bound applies to FO logic on trees!

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- For vertex cover, neighborhood diversity, max-leaf [L. '10]
- For twin cover [Ganian '11]
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Predominant idea: Removing isomorphic parts of the graph, when we have too many

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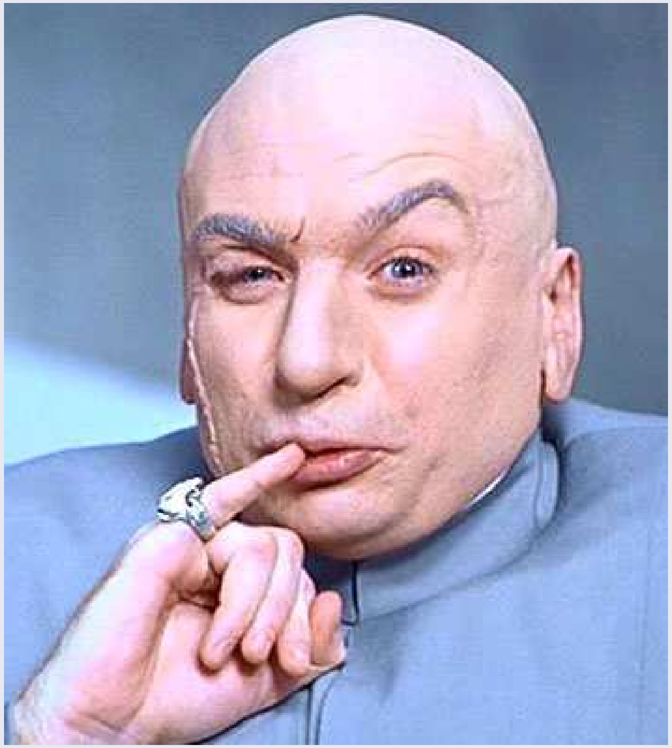
What’s next?

Let's destroy all hope!

- In this talk the pendulum swings again.
- Main goal: prove hardness results even more devastating than Frick& Grohe.
- Motivation: If we know what we can't do, we might find things we can do.



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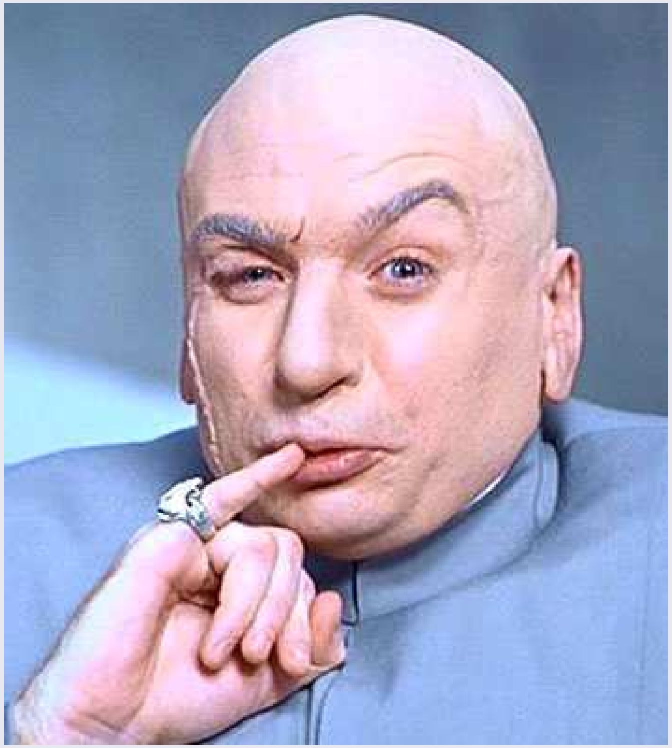

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Today: Three new hardness results.

- Threshold graphs
- Paths
- Bounded-height trees

Threshold Graphs

More background

Theorem:

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- Graphs with clique-width 1 are easy for MSO_1.

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- Graphs with clique-width 1 are easy for MSO_1.

What about clique-width 2?

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A graph is a threshold graph if it can be constructed with the following operations:

- Add a new vertex and connect it to everything.
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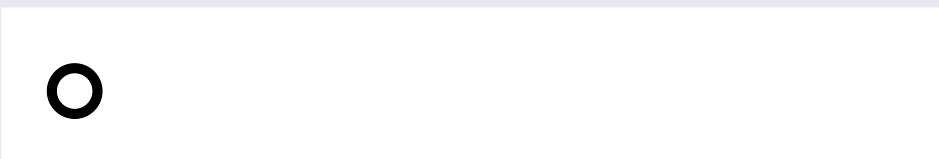

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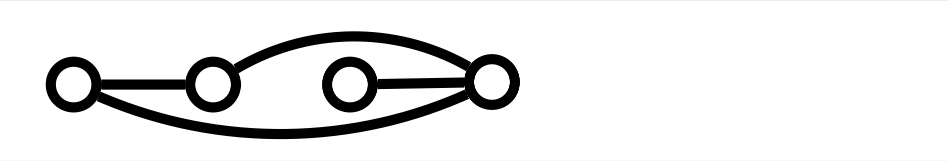

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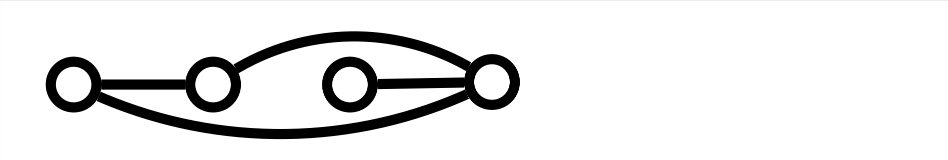

$ujuj$

Thm: Threshold graphs have clique-width 2.

Hardness for threshold graphs

We use the following result of Frick& Grohe:

- There is no elementary-dependence model-checking algorithm for FO logic on binary strings.

Given a string w we construct a threshold graph G

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Thm: There is no elementary-dependence model-checking algorithm for FO logic on threshold graphs.

Consequences

Recall some of the “good” graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
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Recall some of the "good" graph classes we know

- Some are closed under complement (neighborhood diversity, shrub-depth)
- Some are closed under union (tree-depth)
- None are closed under both operations...

Any class of graph closed under both operations must contain threshold graphs.

Paths

Why paths?

Main question:

- Is there an elementary-dependence algorithm for MSO_1 on paths?

Equivalent question:

- Is there an elementary-dependence algorithm for MSO_1 on unary strings?

Why?

- Do Frick and Grohe really need all trees?
- FO is easy on paths.
- MSO is hard on binary strings/colored paths.
- MSO for max-leaf is open!

Why would this be easy?

- MSO on paths = Regular language over unary alphabet
- FO is easy
- Reduction seems impossible...

"Normal" reduction:

- Start with n-variable 3-SAT
- Construct graph G with $|G| = n^c$
- Construct formula ϕ with $|\phi| = \log^* n$
- Prove YES instance $\leftrightarrow G \models \phi$

Problem: New instance would be encodable with $O(\log n)$ bits. We are making a sparse NP-hard language!

How the reduction can work

Key idea: do not use P$\neq$NP but EXP$\neq$NEXP

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Elementary parameter dependence gives EXP=NEXP.

- Formula will be somewhat larger, but still small enough.

Rough sketch

- Start with an NEXP Turing machine, n bits of input. Does it accept?
- The machine runs in time $T = 2^{n^c}$.
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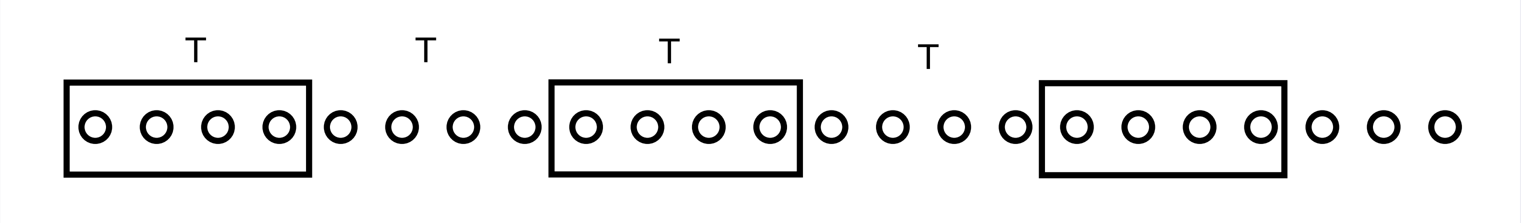

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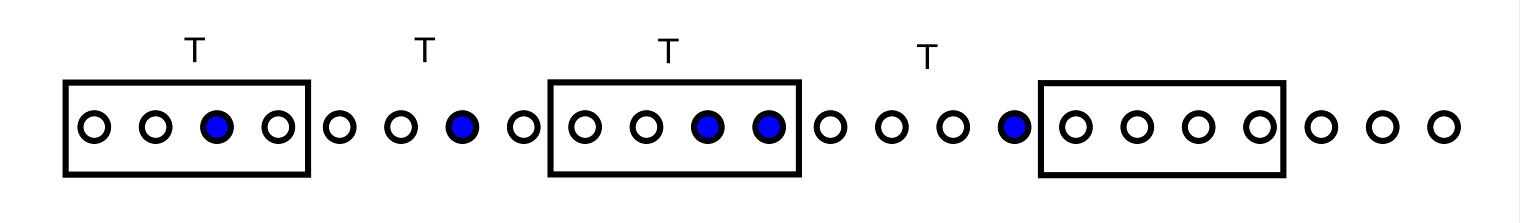

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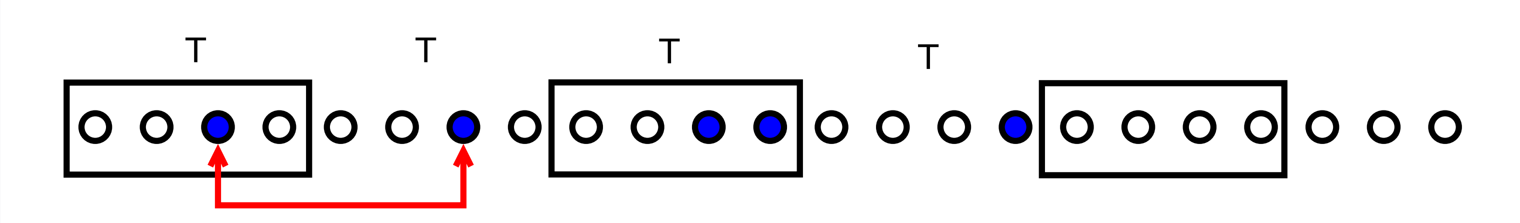

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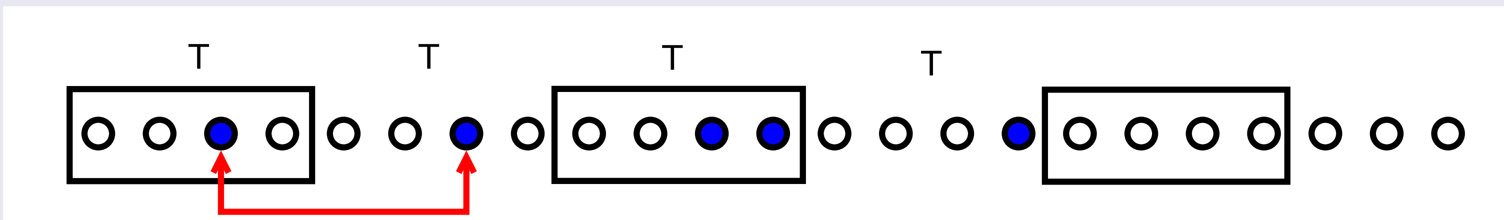

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- We have to do it with $\log^* n$ quantifiers.
- This is possible by encoding counting in binary. . .

Consequences

Unless EXP=NEXP:

- Max-leaf is hard
- Graph classes closed under edge sub-divisions are hard
- Graph classes closed under induced subgraphs with unbounded (dense)* diameter are hard

Trees of bounded height

Why trees of bounded height?

This class of graphs is important for two recent meta-theorems:

- Shrub-depth in "When trees grow low: Shrubs and fast MSO_1" [Ganian et al. MFCS '12]
- Tree-depth in "Faster deciding MSO properties of trees of fixed height, and some consequences" [Gajarský and Hliněný FSTTCS '12]

In both cases the main tool is the following:

> MSO model-checking for q quantifiers on trees of height h colored with t colors can be done in $\exp^{(h+1)}(O(q(t+q))$ time.

Lower bound sketch

Thm: $h + 1$ levels of exponentiation are exactly necessary.

Rough idea: use Frick& Grohe proof for trees, use (few colors) to cut down their height.

- Start from an n-variable 3-SAT instance.
- Construct a tree of height h. Use $t = \log^{(h)}(n)$ colors.
- Construct a formula with $q = O(h)$ quantifiers.
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Argument: an algorithm running in $\exp^{(h+1)}(o(t))$ would run in $2^{o(n)}$ here, disproving ETH.

Conclusions - Open problems

- Three natural barriers to future improvements.
- Paths are probably the toughest to work around.

Future work

- (Uncolored) tree-depth?
- Height of tower for paths?

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Future work

- (Uncolored) tree-depth?
- Height of tower for paths?
- Other logics?!?

Thank you!