KTH Computer Science
and Communication

Homework III, Approximation Algorithms 2010

Due on Tuesday November 9 at 10.15 (hand in at start of lecture or send an email to osven@kth.se). Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students, but solutions should be handed in individually.

1 (45 p) Consider the standard linear programming relaxation of Set Cover (LP 13.2 in Vazirani's book).

1a (15 p) Give a counterexample to the following claim. A set cover instance in which each element is in exactly f sets has a $(1/f)$-integral optimal fractional solution (i.e., in which each set is picked an integral multiple of $1/f$).

Note that when $f = 2$ — the Set Cover instance is simply a Vertex Cover instance — the statement is true as seen in class.

1b (15 p) Let k be a fixed constant, and consider instances of Set Cover where each element can be covered by at most k sets. We proved in class that the integrality gap of the LP is upper bounded by k for these instances. Provide examples to show that this bound is essentially tight.

(This is exercise 15.3 in the book where you can also find a hint if needed.)

1c (15 p) In class we gave a randomized rounding algorithm (see also Section 14.2 in the book). Use similar techniques to give an algorithm that with constant probability returns a collection of sets that cover at least half the elements and has cost at most a constant factor larger than the LP solution.

2 (25 p) In a Vertex Cover instance $G(V, E)$ an edge $\{u, v\} \in E$ equals the constraint that either u or v must be picked. For some applications it is undesirable to pick both so we need to also introduce a second type, called exclusive-or edges, that requires us to pick exactly one of the two incident vertices.

Give a polynomial time algorithm for the generalization of Vertex Cover, where we have both ordinary and exclusive-or edges, that verifies if a solution exists and if it exists returns a 2-approximate solution.

3 (35 p) A car manufacturer wants to create a car consisting of a set $\mathcal{J} = \{1, \ldots, n\}$ of parts. To his disposition he has a set $\mathcal{M} = \{1, \ldots, m\}$ of happy workers. The time it takes for worker $i \in \mathcal{M}$ to construct part $j \in \mathcal{J}$, denoted by p_{ij} is either ∞ if he does not have the required qualifications or otherwise a time $p_j \geq 0$ that only depends on the part. Although the workers are happy they do not work for free: the manufacturer has to pay $c_{ij} \geq 0$ for worker i to construct part j.

The manufacturer has read Section 17 of Vazirani's book and knows a good way to obtain a lower bound T^* on the time needed to construct all parts:

1. Verify a guess T of the minimum time needed by solving $LP(T)$:

$$\min \sum_{i \in \mathcal{M}, j \in \mathcal{J}} x_{ij} \cdot c_{ij}$$

$$\begin{aligned} &\textstyle\sum_{i \in \mathcal{M}} x_{ij} = 1, && j \in \mathcal{J} \\ &\textstyle\sum_{j \in \mathcal{J}} x_{ij} p_{ij} \leq T, && i \in \mathcal{M} \\ &x_{ij} \geq 0, && i \in \mathcal{M}, j \in \mathcal{J} \end{aligned}$$

2. Do binary search to find $T^* = \min\{T : LP(T) \text{ is feasible and } T \geq \max_{j \in \mathcal{J}} p_j\}$.

Your task is to help the manufacturer by giving a polynomial time algorithm that returns a solution satisfying:

1. the time spent to construct all parts is at most $T^* + \max_{j \in \mathcal{J}} p_j$;

2. the cost is no greater than the cost of $LP(T^*)$.

Hint: Let x^* be the vertex achieving the optimum solution to $LP(T^*)$. Now consider the bipartite graph with vertex set $\mathcal{J} \cup \mathcal{M}$ and an edge between a part j and a worker i if $1 > x^*_{ij} > 0$. First, show that this graph is a forest where only workers are leaves. Second, use this fact to assign the parts that were not assigned by the LP.