KTH Computer Science and Communication

Homework VI, Approximation Algorithms 2010

Due on Tuesday December 21 at 10.15. Email the solution to cenny@nada.kth.se or leave it in Ola Svensson's post box. The post boxes/trays can be found around NADA's dining area on Floor 4. Solutions to many homework problems, including problems on this set, are available on the Internet, either in exactly the same formulation or with some minor perturbation. It is *not acceptable* to copy such solutions. It is hard to make strict rules on what information from the Internet you may use and hence whenever in doubt contact Ola Svensson. You are, however, allowed to discuss problems in groups with up to three students, but solutions should be handed in individually.

1 (60 p) In this exercise, we will consider a problem which we call the Snowball-War Problem. We are given a set of kids C that we wish to divide up into two teams, T_1 and T_2. For such a division, everyone in T_1 throws at everyone in T_2, and vice versa. If a kid $c \in T_1$ gets to throw snowballs at a kid $c' \in T_2$, and hence c' at c, we say that this induces a combined fun of $f(c, c')$ (which is equal to $f(c', c)$). The task is to divide up the participants into two teams so as to maximize the total fun of kids that gets to smash each other with frozen H_2O.

1a (30 p) Consider additionally being given a set of pairs L and a set of pairs M. For each pair $\{c, c'\} \in L$, the kids c and c' *must* be in the same team. Similarly, for each pair $\{c, c'\} \in M$, the kids c and c' *must* be in different teams. The goal is to maximize the fun induced by the team division under these conditions.

Adapt an approximation algorithm we have seen in the course to this problem.

(This is Problem 26.12 in Vazirani's book.)

1b (30 p) Assume that instead of having two teams in the Snowball-War Problem, we have k teams for an arbitrary fixed constant k. Naturally, a kid gets to throw snowballs at any kid not in the same team.

Present a real-valued quadratic program for this problem whose optimal value is the same as the optimal solution. Furthermore, the program has to be strict, i.e. contain only constant or quadratic terms. Also present its relaxation to a vector program. You do not have to invent or analyze a rounding of this vector program.

(This is Problem 26.11 in Vazirani's book.)

2 (50 p) There are n helpers of Santa Claus with heights $h_1, \ldots, h_n$. We know that no two helpers have the same height in centimetres. For some triplets (i, j, k) of the helpers you have been told that h_j is between h_i and h_k, i.e. $h_i < h_j < h_k$ or $h_k < h_j < h_i$. The helpers have bad memory, so the information may not be consistent. Given only the set of triplets T, the problem is to find heights of the helpers that maximize the number of satisfied triplets. Note that there are many solutions. For example if $h_1, h_2, \ldots, h_n$ is good then also $h_1 + 1, h_2 + 1, \ldots, h_n + 1$ is good.

Santa Claus developed the following quadratic program, which is feasible if and only if it is possible to satisfy all triplets.

$$\begin{aligned} (h_i - h_j)^2 &\geq 1, \quad \text{for all } i, j \text{ such that } i \neq j \\ (h_i - h_j)(h_k - h_j) &\leq 0, \quad \text{for all } (i, j, k) \in T \end{aligned}$$

Then he translated it into a vector program by substituting h_i by $\mathbf{v}_i$ and multiplication by dot product.

2a (20 p) Give an instance where the above vector program is satisfiable but the instance itself is not satisfiable.

(This is Problem 26.14.5 in Vazirani's book).

Hint: What should be the angle between the vectors $\mathbf{v}_i - \mathbf{v}_j$ and $\mathbf{v}_k - \mathbf{v}_j$?

2b (30 p, bonus) Select $\mathbf{r}$ to be a random unit vector. Consider the random ordering obtained by sorting $\mathbf{r}^T\mathbf{v_i}$. Show that, in expectation, this random ordering satisfies at least half of the constraints in T.

(This is Problem 26.14.6 in Vazirani's book).

Hint: Instead of $\mathbf{r}^T\mathbf{v}_i$, consider for a vector $\mathbf{v}_t$ its projection $\hat{\mathbf{v}}_t$ on the the infinite line l given by $\mathbf{r}$. Now for each triplet (i, j, k), project the line l and the $\hat{\mathbf{v}}_t$-vectors on the plane containing $\mathbf{v}_i - \mathbf{v}_j$ and $\mathbf{v}_k - \mathbf{v}_j$. What is the probability that a triplet is satisfied?