Stochastically Convergent Localization of Objects by Mobile Sensors and Actively Controllable Relative Sensor-Object Pose

Adrian N. Bishop and Patric Jensfelt

***Abstract*— The problem of object (network) localization using a mobile sensor is examined in this paper. Specifically, we consider a set of stationary *objects* located in the plane and a single mobile nonholonomic *sensor* tasked at estimating their relative position from range and bearing measurements. We derive a coordinate transform and a relative sensor-object motion model that leads to a novel problem formulation where the measurements are linear in the object positions. We then apply an extended Kalman filter-like algorithm to the estimation problem. Using stochastic calculus we provide an analysis of the convergence properties of the filter. We then illustrate that it is possible to steer the mobile sensor to achieve a relative sensor-object pose using a continuous control law. This last fact is significant since we circumvent Brockett's theorem and control the relative sensor-source pose using a simple controller.**

I. Introduction

This paper considers the problem of *object* localization using a mobile *sensor* taking relative range and bearing measurements [1], [2]. Furthermore, we consider the problem of actively steering the mobile sensor to achieve a desired relative sensor-object pose with respect to individual objects. The term *object* can be interpreted loosely and might refer to a transmitting node or a target such as an aircraft or missile etc. Alternatively, an object might refer to an everyday object of interest that is to be manipulated by a mobile autonomous robot in an industrial or home environment.

The idea behind the second (control) problem considered in this paper is that it is often the case that a sensor can better localize a target from a particular position or given a certain relative trajectory [3], [4]. Alternatively, we are interested in the problem of localizing a field of objects which the mobile sensor might then wish to return to and manipulate or analyze from certain relative positions.

For example, consider a robot exploring an unknown environment and tasked at localizing a specified class of objects relative to its current position. Following a period of localization, the robot might be asked to return to a particular object and take visual pictures of the object from certain relative positions; i.e. from a certain distance with a certain relative viewing angle [5]. Alternatively, the robot may be required to return to a particular object in order to manipulate or grasp the object for analysis [6] and this task requires a specified relative robot-object pose. It is these sort of scenarios which motivate the formulations and algorithms considered in this paper.

A.N. Bishop and P. Jensfelt are with the Centre for Autonomous Systems, KTH, Stockholm, Sweden. This work was supported by the Centre for Autonomous Systems (CAS) and the EU FP7 project "CogX".

A. Contributions of this Paper

The contributions of this paper are related to the polar-like nature of the derived problem formulation and the rigorous convergence analysis provided. Firstly, we introduce the relative sensor-object dynamic model (given a unicycle robot) in polar coordinates which leads to a linear measurement equation. We then outline an extended Kalman filter (EKF) algorithm that can be used to estimate the relative object positions. We rigorously analyze the convergence of the filter and illustrate a condition which will guarantee the mean-square error exponential convergences to a bounded steady state value. The problem formulation for object localization introduced in this paper is a very natural representation which leads to improved estimation performance.

Following the filter analysis we outline the problem of actively steering the unicycle robot to achieve a desired (relative in this case) pose with respect to an object of interest. We derive a simple continuous control law for the sensor's translational and angular velocities that will steer it to a desired relative distance and angle with respect to the object (or an estimate of the object) position. The polar problem formulation advocated in this paper provides a natural representation of this control problem and simplifies the controller design. To achieve a desired unicycle sensor pose in a global Cartesian framework is non-trivial. Moreover, the relative object-sensor distance and the relative object sensor angle is a more natural representation of the desired (intuitive) control objective.

Thus, the overall concept of object localization and active sensor-object pose control (for localization and viewing the object) is naturally derived in a simple way in this paper. This contribution is significant and aims to highlight the benefits of seeking alternative coordinate systems as a means of simplifying certain nonlinear problems in robotics, localization and multi-agent systems control.

II. Preliminaries

Consider a mobile sensor with a state description $\mathrm{s} = [x\ y\ \phi]^{\mathrm{T}} \in \{\mathbb{R}^2 \times \mathbb{SO}(1,\mathbb{R})\}$. Here x and y are the sensor's Cartesian position coordinates and ϕ is the sensor's heading. The sensor dynamics are based on the *unicycle* model,

$$\begin{aligned} \dot{x}_r &= v_r \cos\phi_r \\ \dot{y}_r &= v_r \sin\phi_r \\ \dot{\phi}_r &= w_r \end{aligned} \tag{1}$$

where v is the translational velocity and w is the sensor's angular velocity. Note that there are three state variables in

$\mathbb{R}^2 \times \mathbb{SO}(1,\mathbb{R})$ and only two control inputs. The nonholonomic constraint on the sensor is given by

$$\dot{x}_r \sin\phi_r = \dot{y}_r \cos\phi_r \tag{2}$$

and implies via Brockett's theorem that a desired robot pose $\mathbf{s}^* = [x^* \; y^* \; \phi^*]^{\mathrm{T}}$ can not be asymptotically stabilized using a linear smooth time-invariant control law. We assume the control inputs v and w are known precisely (although we can relax this assumption, we do not do so in this paper).

The environment is populated with a set $\mathcal{V}$ of *objects* or target nodes with $|\mathcal{V}| = n$. Here objects might mean source nodes (e.g. active enemy radars, acoustic sources etc), landmarks or feature points as discussed in the simultaneous localization and mapping literature, or targets such as aircraft, missiles etc. Alternatively, objects might mean everyday objects of interest that are to be manipulated by a mobile autonomous robot in a home/industrial environment.

The Cartesian position of the i^{th} object is denoted by $\mathbf{p}_i = [x_i \; y_i]^{\mathrm{T}} \in \mathbb{R}^2$. The objects are stationary in this case and represent the *map* of the environment which is to be estimated by the mobile sensor. At some time t the sensor can sense a subset $\mathcal{G}(t) \subseteq \mathcal{V}$ of landmarks. At time t the true measurements of object i are given by

$$\begin{aligned} d_i &= \sqrt{(x_i - x)^2 + (y_i - y)^2} \\ \vartheta_i &= \theta_i - \phi = \arctan\left(\frac{y_i - y}{x_i - x}\right) - \phi \\ &\quad \forall i \in \mathcal{G}(t) \end{aligned} \tag{3}$$

where $\vartheta_i = \theta_i - \phi$ is the relative bearing to the i^{th} object in the sensor's internal Cartesian coordinate system, i.e. the Cartesian coordinate system rotated by the sensor's heading. Let $\mathbf{z} = [\mathbf{s}_r \; \mathbf{p}_1 \; \dots \; \mathbf{p}_n]^{\mathrm{T}}$. The measurements are typically corrupted by a noise process $\mathbf{n}(t)$ and thus we can obtain the measurement equation

$$d\mathbf{y}(t) \triangleq \psi(t)dt = h(\mathbf{z})dt + \mathbf{E}(t)d\mathbf{n}(t) \tag{4}$$

in continuous-time. Here, $\mathbf{n}(t)$ is a zero-mean Weiner process and $\mathbf{E}(t)$ is a measurement noise weighting matrix that can be dependent on the true state. For example, it might be true that the noise present in the range measurements is a fraction of the true range. The measurements and robot dynamics are nonlinear in the chosen coordinate system.

III. Localization of Objects in Polar Coordinates

One contribution of this paper is a novel localization analysis that takes advantage of the polar-like nature of the relative range and bearing measurements. There is a long history in the bearing-only target tracking literature [1], [7] of working in variants of polar coordinate systems. Here, we derive a relative sensor-object motion model and then formulate an estimation problem that involves linear measurements (which can significantly improve the performance of the EKF as noted in many different example problems [1], [7]).

Recall the true measurements taken by the mobile sensor are given by

$$\begin{aligned} d_i &= \sqrt{(x_i - x)^2 + (y_i - y)^2} \\ \vartheta_i &= \theta_i - \phi = \arctan\left(\frac{y_i - y}{x_i - x}\right) - \phi \\ &\quad \forall i \in \mathcal{G}(t) \end{aligned} \tag{5}$$

where the state $\mathbf{s} = [x \; y \; \phi]^{\mathrm{T}}$ of the sensor and the position of the objects $\mathbf{p}_i = [x_i \; y_i]^{\mathrm{T}} \in \mathbb{R}^2$ are in some external (non-relative) coordinate system. The measurements are nonlinear in the first two components of $\mathbf{s}$ and in $\mathbf{p}_i$, $\forall i$.

Now define the following state variable $\mathbf{r}_i = [d_i \; \vartheta_i]^{\mathrm{T}}$ with $d_i \in (0, \infty)$ and $\vartheta_i \in [-\pi, \pi)$. We will always assume that $d_i \neq 0$ for both theoretical and very practical reasons. The augmented state variable in this section is given by $\mathbf{z} = [\mathbf{r}_1 \; \dots \; \mathbf{r}_n]^{\mathrm{T}}$ and encompasses the relative sensor-object position for all objects in the set. In practice the state $\mathbf{z}$ can be augmented online when each new object is sensed.

The measurements (5) are linear in $\mathbf{r}_i$ or more generally in $\mathbf{z} = [\mathbf{r}_1 \; \dots \; \mathbf{r}_n]^{\mathrm{T}}$ and are given by the continuous-time measurement equation

$$d\mathbf{y}(t) \triangleq \psi(t)dt = \mathbf{H}(\mathcal{G}(t))\mathbf{z}dt + \mathbf{E}(t)d\mathbf{n}(t) \tag{6}$$

where $\mathbf{E}(t)$ is not required to be independent of $\mathbf{z}$ (as discussed previously). Here $\mathbf{H}(\mathcal{G}(t))$ is a time-varying linear matrix which is dependent only on the set $\mathcal{G}(t)$ of currently sensed landmarks. For example, if all of the landmarks are sensed and the state variable $\mathbf{z}$ is ordered appropriately, then $\mathbf{H}$ would be the identity matrix.

Consider again a robot that obeys the unicycle model (1) in $\mathbb{R}^2 \times \mathbb{SO}(1,\mathbb{R})$. Then we can write down the following differential equation for the dynamics of $\mathbf{r}_i$,

$$\begin{aligned} \dot{d}_i &= -v \cos\vartheta_i \\ \dot{\vartheta}_i &= \frac{v}{d_i} \sin\vartheta_i - w \end{aligned} \tag{7}$$

which is nonlinear in $\mathbf{r}_i$. Note also that d_i must be bounded away from zero here for technical reasons (although practically this is also logical). Again we assume v and w are known precisely.

A. On the Observability of the Polar Localization Problem and the Convergence of the EKF-Based Algorithm

In this subsection we will examine and prove a number of results regarding the convergence of an EKF-like algorithm for estimating the relative object state variable.

1) Error Free Measurements and Dynamics: We consider first the observability properties of the state $\mathbf{z} = [\mathbf{r}_1 \; \dots \; \mathbf{r}_n]^{\mathrm{T}}$ with $\mathbf{r}_i = [d_i \; \vartheta_i]^{\mathrm{T}}$ evolving according to (7). We also assume error free measurements of the form $\psi(t) = \mathbf{H}(\mathcal{G}(t))\mathbf{z}(t)$.

Corollary 1: Assume the robot-landmark dynamics and the measurements are deterministic and error free. The state $\mathbf{r}_i(s) = [d_i(s) \; \vartheta_i(s)]^{\mathrm{T}}$ for some $i \in \mathcal{V}$ and for $s \geq \tau$ or $s < \tau$ can be calculated at any time $t \geq \tau$ if and only if $\mathcal{G}(\tau) \cap \mathbf{r}_i(\tau) \neq \emptyset$ for some instant τ.

The fact that Corollary 1 is true is not surprising but is provided for completeness.

2) Error Free Dynamics and Noisy Measurements: A natural extension to the above result concerns the behavior of an estimate $\widehat{\mathbf{z}}$ of $\mathbf{z}$ when the dynamics of the state $\mathbf{r}_i = [d_i \ \vartheta_i]^{\mathrm{T}}$ are error free and deterministic but the measurements

$$d\mathbf{y}(t) = \mathbf{H}(\mathcal{G}(t))\mathbf{z}dt + \mathbf{E}(t)d\mathbf{n}(t) \tag{8}$$

are corrupted by an additive Weiner process. Naturally, the behavior of any state estimate $\widehat{\mathbf{z}}$ depends on the particular estimator and thus let us consider an estimator of the form

$$d\widehat{\mathbf{z}} = f(\widehat{\mathbf{z}}, v, w)dt + \mathbf{K}(t)\left(d\mathbf{y}(t) - \mathbf{H}(\mathcal{G}(t))\widehat{\mathbf{z}}dt\right) \tag{9}$$

where the function $f_i(\cdot)$ that captures the dynamics of the subspace $\mathbf{r}_i = [d_i \ \vartheta_i]^{\mathrm{T}}$ is given by

$$f_i(\widehat{\mathbf{z}}, v, w) = \begin{bmatrix} -v\cos\widehat{\vartheta}_i \\ \frac{v}{d_i}\sin\widehat{\vartheta}_i - w \end{bmatrix} \tag{10}$$

where v and w are again considered as deterministic control inputs with no errors. The function $f(\cdot)$ is thus a vertical concatenation of the $f_i(\cdot)$. The gain $\mathbf{K}(t)$ is given by

$$\mathbf{K}(t) = \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(\mathcal{G}(t))\mathbf{R}^{-1}(t) \tag{11}$$

and $\mathbf{P}(t)$ is the solution to the following Riccati differential equation

$$\begin{aligned} d\mathbf{P}(t) &= \left[\mathbf{A}(t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{A}^{\mathrm{T}}(t) + \mathbf{Q}(t)\right]dt - \\ &\quad \mathbf{P}(t)\mathbf{H}^{\mathrm{T}}(\mathcal{G}(t))\mathbf{R}^{-1}(t)\mathbf{H}(\mathcal{G}(t))\mathbf{P}(t)dt \end{aligned} \tag{12}$$

where $\mathbf{Q}$ and $\mathbf{R}$ are positive-definite tuning matrices. Note that $\mathbf{A}(t)$ is the Jacobian of $f(\cdot)$ evaluated at $\widehat{\mathbf{z}}$. The Jacobian $\mathbf{A}_i(t)$ of $f_i(\cdot)$ is given by

$$\mathbf{A}_i(t) = \begin{bmatrix} 0 & v\sin\widehat{\vartheta}_i \\ -\frac{v}{\widehat{d}_i^2}\sin\widehat{\vartheta}_i & \frac{v_r}{\widehat{d}_i}\cos\widehat{\vartheta}_i \end{bmatrix} \tag{13}$$

and is evaluated at $\widehat{\mathbf{r}}_i$ and is dependent on v. Note the estimation error $\boldsymbol{\zeta} = \mathbf{z} - \widehat{\mathbf{z}}$ evolves according to

$$\begin{aligned} d\boldsymbol{\zeta} &= \left(\mathbf{A}(t) - \mathbf{K}(t)\mathbf{H}(\mathcal{G}(t))\right)\boldsymbol{\zeta}dt + \\ &\quad \varrho(\mathbf{z}, \widehat{\mathbf{z}}, v, w)dt - \mathbf{K}(t)\mathbf{N}(t)d\mathbf{n}(t) \end{aligned} \tag{14}$$

where we have used the following Taylor expansion of $f(\cdot)$ about the estimate $\widehat{\mathbf{z}}$,

$$f(\mathbf{z}, v, w) - f(\widehat{\mathbf{z}}, v, w) = \mathbf{A}(t)\left(\mathbf{z} - \widehat{\mathbf{z}}\right) + \varrho(\mathbf{z}, \widehat{\mathbf{z}}, v, w) \tag{15}$$

where $\varrho(\mathbf{z}, \widehat{\mathbf{z}}, v, w)$ accounts for the higher order terms. Recall that $\mathbf{r}_i = [d_i \ \vartheta_i]^{\mathrm{T}}$ with $d_i \in (0, \infty)$ and $\vartheta_i \in [-\pi, \pi)$ for all t. Then it is clear that the following bound holds

$$\|\mathbf{A}(t)\| = \overline{a} < \infty \tag{16}$$

for all t where for any time-varying matrix $\mathbf{M}(t)$ we assume the following

$$\|\mathbf{M}(t)\| = \sup\{\|\mathbf{M}(t)\| : m_{ij} \in \mathbb{M}_{ij} \subseteq \mathbb{R}\} \tag{17}$$

for all t and for some norm $\|\cdot\|$. At this point we make the following assumptions.

Assumption 1: The translational velocity of the robot $v(t)$ is uppperbounded in any arbitrary coordinate scale such that $v(t) \leq \overline{v}$ for all t. For simplicity we also assume that $v(t) > 0$ for all t. Now it follows that there exists a temporal coordinate scale such that $v(t) \leq 1$ for all t.

Assumption 2: The relative distance between the robot and the i^{th} landmark at time t belongs to the space $d_i(t) \in (0, \infty)$ in any arbitrarily chosen coordinate scale. There exists a spatial coordinate scale such that for all t we have $d_i(t) \in [1, \infty)$.

Assumptions 1 and 2 are weak (actually notational) and can almost surely be satisfied in practice (i.e. by finding explicit spatial and temporal scales). The case of $v = 0$ is trivially obtained from the subsequent results. For simplicity we also assume the following.

Assumption 3: For all t we have $\widehat{\mathbf{r}}_i(t) = [\widehat{d}_i(t) \ \widehat{\vartheta}_i(t)]^{\mathrm{T}}$ with $\widehat{d}_i(t) \in [1, \infty)$ and $\widehat{\vartheta}_i(t) \in [-\pi, \pi)$.

Assumption 3 calls for the state estimate components to be restricted to the assumed true global state space. Finally, we make the following assumption on the design parameters.

Assumption 4: The following $\mathbf{Q}(t) \geq \underline{q}\mathbf{I}$, $\mathbf{R}(t) \geq \underline{r}\mathbf{I}$ and $\mathbf{P}(t_0) \geq p_0\mathbf{I}$ are given for some $\underline{q}, \underline{r}, p_0 > 0$ such that $\|\mathbf{Q}(t)\| \geq \underline{q}$ and $\|\mathbf{R}(t)\| \geq \underline{r}$. Moreover, $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ are chosen to be bounded by $\|\mathbf{Q}(t)\| \leq \overline{q} < \infty$ and $\|\mathbf{R}(t)\| \leq \overline{r} < \infty$ for all t. Also, we have $\|\mathbf{E}(t)\| \leq \overline{e} < \infty$ with $\mathbf{E}(t) \geq \underline{e}\mathbf{I}$.

Clearly, Assumption 4 is standard. We will also need the following lemma concerning the growth of $\varrho(\mathbf{z}, \widehat{\mathbf{z}}, v_r, w_r)$.

Lemma 1: The following inequality holds

$$\begin{aligned} \|\varrho(\mathbf{z}, \widehat{\mathbf{z}}, v_r, w_r)\| &= \|f(\mathbf{z}, \cdot) - f(\widehat{\mathbf{z}}, \cdot) - \mathbf{A}(t)\left(\mathbf{z} - \widehat{\mathbf{z}}\right)\| \\ &\leq 2\overline{a}\|\boldsymbol{\zeta}\| \end{aligned} \tag{18}$$

for $|\mathcal{V}| = n$ with probability 1 when Assumptions 1-4 hold.

Proof: The proof is trivial and follows from (16) and the triangle inequality. ■

Note also that $\varrho(\mathbf{z}, \widehat{\mathbf{z}}, v_r, w_r) = 0$ when $\boldsymbol{\zeta}(t) = 0$. We assume the initial estimation error $\boldsymbol{\zeta}(t_0)$ belongs to the set

$$\boldsymbol{\zeta}(t_0) \in \{\boldsymbol{\eta} \in \{[0, \infty) \times [-\pi, \pi)\} : \|\boldsymbol{\zeta}(t_0)\| \leq d\} \tag{19}$$

for some constant $d < \infty$. We also assume initially that $\mathcal{G}(t) = \mathcal{V}$ for all $t > t_0$ and thus the error propagates according to (14) with (for simplicity) $\mathbf{H}(\mathcal{G}(t)) = \mathbf{I}$ for all t. It is common to assume a full measurement vector when performing such an analysis [8].

Lemma 2: Suppose Assumptions 1-4 hold. Then the state estimate covariance $\mathbf{P}(t)$ is bounded by

$$0 < \underline{p} \leq \|\mathbf{P}(t)\| \leq \overline{p} < \infty \tag{20}$$

for all $t > t_0$ and where

$$\overline{p} \triangleq \left(\|\mathbf{P}(t_0)\| + \frac{\|\mathbf{Q}(t)\| + \|\mathbf{R}(t)\|\|\mathbf{A}(t)\|^2}{2\kappa}\right) \tag{21}$$

and where $\mathbf{\Lambda}$ is chosen such that

$$\boldsymbol{\eta}^{\mathrm{T}}\left(\mathbf{A}(t)+\mathbf{\Lambda}(t)\right)\boldsymbol{\eta} \leq -\kappa\|\boldsymbol{\eta}\|^2 \tag{22}$$

is satisfied for all $\boldsymbol{\eta} \in \mathbf{R}^2$ with $\kappa > 0$.

Proof: The upper bound can be obtained by considering the following time-varying linear control system

$$-\dot{\mathbf{q}} = \mathbf{A}(t)\mathbf{q} + \mathbf{u} \tag{23}$$

with a boundary $\mathbf{q}(T) = \mathbf{q}_T$ for some $\infty \geq T > 0$ and with controllability Grammian

$$\mathcal{C}(t+\tau, t) = \int_t^{t+\tau} \mathbf{\Phi}(t+\tau,t)\mathbf{\Phi}^{\mathrm{T}}(t+\tau,t)dt \tag{24}$$

where $\mathbf{\Phi}(t+\tau, t)$ is the fundamental matrix with $\mathbf{\Phi}(t,t) = \mathbf{I}$. Clearly, the system (23) is uniformly completely controllable. Consider the following cost function

$$\mathcal{J}(t,\tau,\mathbf{q},\mathbf{u}) = \mathcal{B}(t_0,\mathbf{q}(t_0)) + \int_{t_0}^{T}\left(\mathbf{q}^{\mathrm{T}}\mathbf{Q}\mathbf{q} + \mathbf{u}^{\mathrm{T}}\mathbf{R}\mathbf{u}\right)dt \tag{25}$$

and value function $\mathcal{B}(t, \mathbf{q}(t)) = \mathbf{q}^{\mathrm{T}}(t)\mathbf{P}(t)\mathbf{q}(t)$. Let the control input equal $\mathbf{u}(t) = \mathbf{\Lambda}(t)\mathbf{q}$ for some continuous bounded matrix $\mathbf{\Lambda}(t)$ such that $-\dot{\mathbf{q}} = (\mathbf{A}(t) + \mathbf{\Lambda}(t))\mathbf{q}$. Note now that

$$\begin{aligned}\mathcal{B}(T,\mathbf{q}(T)) &= \mathbf{q}^{\mathrm{T}}(T)\mathbf{P}(T)\mathbf{q}(T)\\ &\leq \mathcal{B}(t_0,\mathbf{q}(t_0)) +\\ &\quad \int_{t_0}^{T}\mathbf{q}^{\mathrm{T}}\left(\mathbf{Q}+\mathbf{\Lambda}^{\mathrm{T}}(t)\mathbf{R}\mathbf{\Lambda}(t)\right)\mathbf{q}dt\end{aligned} \tag{26}$$

Solving $-\dot{\mathbf{q}} = (\mathbf{A}(t) + \mathbf{\Lambda}(t))\mathbf{q}$ for $\mathbf{q}(T)$ implies that

$$\begin{aligned}\|\mathbf{q}(T)\|^2 &= \|\mathbf{q}_T\|^2 = \|\mathbf{q}(t_0)\|^2 -\\ &\quad 2\int_{t_0}^{T}\mathbf{q}^{\mathrm{T}}\left(\mathbf{A}(t)+\mathbf{\Lambda}(t)\right)\mathbf{q}\; dt\end{aligned} \tag{27}$$

and (22) implies $\|\mathbf{q}(t_0)\|^2 \leq \|\mathbf{q}_T\|^2$ and $\int_{t_0}^{T}\mathbf{q}^{\mathrm{T}}\mathbf{q}\; dt \leq \frac{\|\mathbf{q}_T\|^2}{2\kappa}$. Using this with (26) leads to the upper-bound. ∎

Note that $\|\mathbf{P}(t)\|$ is bounded above by a constant independent of the time $t > t_0$. Part of Lemma 2 follows from a theorem given in [9]. The condition (22) calls for the system pair $\mathbf{A}(t)$ and $\mathbf{H}(\mathcal{G}(t))$ to be uniformly detectable. In our case we know that the system is observable (which implies detectability [9], [10]). As such, a suitable matrix $\mathbf{\Lambda}(t)$ exists with probability one.

Theorem 1: Consider the system (14) with an initial condition (19) and $\mathbf{H}(\mathcal{G}(t)) = \mathbf{I}$. Suppose that Assumptions 1-4 hold. If $\|\mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t) + \mathbf{R}^{-1}(t)\|\underline{p} > \frac{4\overline{ap}}{\underline{p}}$ then the estimation error is bounded above with

$$\mathcal{E}\{\|\boldsymbol{\zeta}(t)\|^2\} \leq \max\left\{\frac{n\overline{p}^2\overline{e}^2}{2\gamma\underline{r}^2}\;,\;\frac{\overline{p}}{\underline{p}}\|\boldsymbol{\zeta}(t_0)\|^2\right\} \tag{28}$$

where $\gamma = \|\mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t) + \mathbf{R}^{-1}(t)\|\underline{p} - 4\overline{ap}/\underline{p}$ and the error $\mathcal{E}\{\|\boldsymbol{\zeta}(t)\|^2\}$ as $t \to \infty$ is bounded by $\frac{n\overline{p}^2\overline{e}^2}{2\gamma\underline{r}^2}$.

Proof: The error system (14) can be thought of as a linear system with a nonlinear perturbation being driven by a zero-mean Weiner process. Let $\mathcal{B}(t,\boldsymbol{\zeta}(t)) = \boldsymbol{\zeta}^{\mathrm{T}}(t)\mathbf{P}^{-1}(t)\boldsymbol{\zeta}(t) > 0$ and note that

$$\begin{aligned}d\mathcal{B} &= \left[\frac{\partial\mathcal{B}}{\partial t} + \frac{\partial\mathcal{B}}{\partial\boldsymbol{\zeta}}\left(\mathbf{A}(t)-\mathbf{K}(t)\right)\boldsymbol{\zeta}\right]dt +\\ &\quad \frac{\partial\mathcal{B}}{\partial\boldsymbol{\zeta}}\varrho(\mathbf{z},\widehat{\mathbf{z}},v_r,w_r)dt +\\ &\quad \frac{1}{2}\mathrm{tr}\left(\mathrm{hess}(\mathcal{B})\mathbf{K}(t)\mathbf{E}(t)\mathbf{E}^{\mathrm{T}}(t)\mathbf{K}^{\mathrm{T}}(t)\right)dt -\\ &\quad \frac{\partial\mathcal{B}}{\partial\boldsymbol{\zeta}}\mathbf{K}(t)\mathbf{E}(t)d\mathbf{n}\\ d\mathcal{B} &= \left[\frac{\partial\mathcal{B}}{\partial t} + \mathcal{L}\mathcal{B}\right]dt - \frac{\partial\mathcal{B}}{\partial\boldsymbol{\zeta}}\mathbf{K}(t)\mathbf{E}(t)d\mathbf{n}\end{aligned} \tag{29}$$

using Ito's differential formula and where $\mathcal{L}$ is the Kolmogorov backward operator, $\mathrm{hess}(\cdot)$ denotes the Hessian operator and $\mathrm{tr}(\cdot)$ denotes the matrix trace. Evaluating the terms and re-arranging leads to

$$\begin{aligned}d\mathcal{B} &= \left[\boldsymbol{\zeta}^{\mathrm{T}}\left[-\mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t) - \mathbf{R}^{-1}(t)\right]\boldsymbol{\zeta}\right]dt +\\ &\quad 2\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{P}^{-1}(t)\varrho(\mathbf{z},\widehat{\mathbf{z}},v_r,w_r)dt +\\ &\quad \frac{1}{2}\mathrm{tr}\left(\mathbf{R}^{-1}(t)\mathbf{E}(t)\mathbf{E}^{\mathrm{T}}(t)\mathbf{R}^{-\mathrm{T}}(t)\mathbf{P}^{\mathrm{T}}(t)\right)dt -\\ &\quad 2\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{R}^{-1}(t)d\mathbf{n}\\ &\leq \left[-\alpha\|\boldsymbol{\zeta}\|^2 + \frac{4\overline{a}}{\underline{p}}\|\boldsymbol{\zeta}\|^2 + \frac{n\overline{pe}^2}{2\underline{r}^2}\right]dt -\\ &\quad 2\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{R}^{-1}(t)d\mathbf{n}\end{aligned} \tag{30}$$

where we have explicitly employed Lemma 1 and Lemma 2 and where

$$\alpha = \|\mathbf{P}^{-1}(t)\mathbf{Q}(t)\mathbf{P}^{-1}(t) + \mathbf{R}^{-1}(t)\| \tag{31}$$

Clearly we have $\overline{p}^{-1}\|\boldsymbol{\zeta}\|^2 \leq \mathcal{B}(t,\boldsymbol{\zeta}(t)) \leq \underline{p}^{-1}\|\boldsymbol{\zeta}\|^2$ such that some simple algebra implies that

$$\begin{aligned}d\mathcal{B} &\leq -\left(\alpha\underline{p} - \frac{4\overline{ap}}{\underline{p}}\right)\mathcal{B}dt + \frac{n\overline{pe}^2}{2\underline{r}^2}dt -\\ &\quad 2\boldsymbol{\zeta}^{\mathrm{T}}\mathbf{R}^{-1}(t)d\mathbf{n}\\ \mathcal{B} &\leq \mathcal{B}(t_0,\boldsymbol{\zeta}(t_0)) -\\ &\quad \int_{t_0}^{t}\left(\alpha\underline{p} - \frac{4\overline{ap}}{\underline{p}}\right)\mathcal{B}(\tau,\boldsymbol{\zeta}(\tau))d\tau +\\ &\quad \frac{n\overline{pe}^2}{2\underline{r}^2}\int_{t_0}^{t}d\tau - 2\int_{t_0}^{t}\boldsymbol{\zeta}^{\mathrm{T}}(\tau)\mathbf{R}^{-1}(\tau)d\mathbf{n}(\tau)\end{aligned} \tag{32}$$

From the Bellman-Gromwall lemma [11] we have

$$\begin{aligned}\mathcal{B}(t,\boldsymbol{\zeta}(t)) &\leq \mathcal{B}(t_0,\boldsymbol{\zeta}(t_0))\exp\left(-\gamma(t-t_0)\right) +\\ &\quad \frac{n\overline{pe}^2}{2\gamma\underline{r}^2}\left(1-\exp\left(-\gamma(t-t_0)\right)\right) -\\ &\quad 2\int_{t_0}^{t}\boldsymbol{\zeta}^{\mathrm{T}}(\tau)\mathbf{R}^{-1}(\tau)d\mathbf{n}(\tau)\end{aligned} \tag{33}$$

where

$$\gamma = \left(\alpha\underline{p} - 4\overline{ap}/\underline{p}\right) \tag{34}$$

with $\gamma > 0$ if and only if $\alpha\underline{p} > \frac{4\overline{ap}}{\underline{p}}$. Taking the expectation $\mathcal{E}\{\cdot\}$ of both sides of (33) gives

$$\begin{aligned}\mathcal{E}\{\mathcal{B}(t,\boldsymbol{\zeta}(t))\} \quad &\leq \quad \mathcal{B}(t_0,\boldsymbol{\zeta}(t_0))\exp\left(-\gamma(t-t_0)\right) + \\ &\frac{n\overline{pe}^2}{2\gamma\underline{r}^2}\left(1-\exp\left(-\gamma(t-t_0)\right)\right) \quad (35)\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{E}\{\|\boldsymbol{\zeta}(t)\|^2\} \quad &\leq \quad \frac{\overline{p}}{\underline{p}}\|\boldsymbol{\zeta}(t_0)\|^2\exp\left(-\gamma(t-t_0)\right) + \\ &\frac{n\overline{p}^2\overline{e}^2}{2\gamma\underline{r}^2}\left(1-\exp\left(-\gamma(t-t_0)\right)\right) \quad (36)\end{aligned}$$

We then easily find that

$$\mathcal{E}\{\|\boldsymbol{\zeta}(t)\|^2\} \leq \max\left\{\frac{n\overline{p}^2\overline{e}^2}{2\gamma\underline{r}^2}\ ,\ \frac{\overline{p}}{\underline{p}}\|\boldsymbol{\zeta}(t_0)\|^2\right\} \qquad (37)$$

for all t if $\gamma > 0$ and the error $\mathcal{E}\{\|\boldsymbol{\zeta}(t)\|^2\}$ as $t \to \infty$ is bounded by $\frac{n\overline{p}^2\overline{e}^2}{2\gamma\underline{r}^2}$. This completes the proof. ■

Importantly, we have shown under what conditions an EKF-like algorithm will yield an exponentially bounded and converging mean-square estimation error. The asymptotic mean-square estimation error is dependent on the specific robot trajectory but is upper-bounded by $\frac{n\overline{p}^2\overline{e}^2}{\gamma\underline{r}^2}$. Theorem 1 is a significant contribution to the problem of localization using a mobile sensor and is a fundamental result.

Corollary 2: Suppose that Assumptions 1-4 hold and $(\mathbf{A},\mathbf{H}(t))$ *is a uniformly detectable pair (which is guaranteed since* $(\mathbf{A},\mathbf{H}(t))$ *is actually an observable pair). Now if* $\gamma > 0$ *and* $\mathbf{n} \to \mathbf{0}$*, then* $\|\boldsymbol{\zeta}(t)\| \to 0$ *as* $t \to \infty$*.*

That is, as the measurement noise approaches zero, the estimation error will asymptotically (and actually exponentially [9], [12]) converge to zero given the satisfaction of the required conditions; i.e. the EKF as applied in this paper acts as an asymptotic nonlinear observer; e.g. see [7], [9], [12]–[14]. Thus, Corollary 2 and Theorem 1 justify application of the EKF in well-posed scenarios (where the noise is small). We can also derive a result similar to Theorem 1 when process noise (i.e. control input noise) is present. For brevity and due to space limitations, estimator simulations will appear in an extended version of the paper.

IV. Active Sensor-Object Pose Control

We now illustrate a technique to steer the sensor to a desired relative sensor-object pose $\mathbf{t}_i = [d_{ti}\ \vartheta_{ti}]^{\mathrm{T}}$ using a simple continuous control law; e.g. similarly to the formation control problem [15]. This might be desired if the mobile sensor wishes to view (with a visual sensor for example) a particular object $i \in \mathcal{V}$ from a (possibly estimated) distance and viewing angle. Similarly, the mobile sensor might be a robot which must achieve a certain robot-object pose in order to manipulate the object in some manner (due to the physical configuration or constraints of the manipulation device).

Consider the global Cartesian sensor motion equations (1) and a Cartesian representation of the i^{th} object's position. Steering the sensor to achieve a desired distance d_i and a desired relative (viewing) angle ϑ_i with the object is non-trivial since the desired objective is not stated linearly in the sensor state components. Moreover, it would require a discontinuous or time-varying nonlinear control law.

However, consider the relative state $\mathbf{r}_i = [d_i\ \vartheta_i]^{\mathrm{T}}$ and the problem of steering the mobile sensor to a desired relative state $\mathbf{t}_i = [d_{ti}\ \vartheta_{ti}]^{\mathrm{T}}$. Note that the control objective is expressed naturally and the sensor state-object state is linearly related to the objective.

The described polar formulation also has a very attractive property in that we can use Lyupunov techniques to design the stabilizing sensor-object pose control law. Brockett's (negative) theorem is in a sense circumvented (albeit we do not control the robot pose in a global sense) and the practicality is (arguably) increased by considering such a formulation. The controller we outline is continuous and leads to very natural trajectories.

We now outline the control law for v and w that will steer the mobile sensor to have a desired (or target) pose $\mathbf{t}_i = [d_{ti}\ \vartheta_{ti}]^{\mathrm{T}}$ with respect to the estimated state of object i given by $\widehat{\mathbf{r}}_i(\tau) = [\widehat{d}_i(\tau)\ \widehat{\vartheta}_i(\tau)]^{\mathrm{T}}$ at some time τ. The following remark concerns an implicit technical requirement of the controller with respect to the considered estimation problem outlined in the previous section.

Remark 1: We have a state subspace estimate $\widehat{\mathbf{r}}_i(\tau) = [\widehat{d}_i(\tau)\ \widehat{\vartheta}_i(\tau)]^{\mathrm{T}}$ *at some time* τ *as the output from the EKF algorithm discussed in the previous section. Now we can set to zero the Kalman gain* $\mathbf{K}(t)$ *subspace corresponding to the state* $\widehat{\mathbf{r}}_i$ *of object* i *for all* $t > \tau$*. Then we have measurements (or estimates as it so happens) of the relative sensor object pose* $\widehat{\mathbf{r}}_i(t) = [\widehat{d}_i(t)\ \widehat{\vartheta}_i(t)]^{\mathrm{T}}$ *for all* $t > \tau$ *that are not affected by a stochastic process* $\forall t > \tau$*. For example, if the sensor does not move such that* $v = 0$ *and* $w = 0$ *then* $\widehat{\mathbf{r}}_i(t) = [\widehat{d}_i(t)\ \widehat{\vartheta}_i(t)]^{\mathrm{T}}$ *for all* $t > \tau$ *is constant. This does not necessarily occur when the Kalman gain* $\mathbf{K}(t)$ *subspace corresponding to the state* $\widehat{\mathbf{r}}_i$ *is non-zero and we are taking measurements of object* i*.*

Thus we want the control error

$$\boldsymbol{\delta}_{ti}(t) = \mathbf{t}_i - \widehat{\mathbf{r}}_i(t) = [d_{ti}\ \vartheta_{ti}]^{\mathrm{T}} - [\widehat{d}_i(t)\ \widehat{\vartheta}_i(t)]^{\mathrm{T}},\ \ t > \tau \quad (38)$$

to be minimized to zero where $\widehat{\mathbf{r}}_i(t)$ is the subspace output of the EKF-like algorithm given that we have set to zero the Kalman gain $\mathbf{K}(t)$ subspace corresponding to the state $\widehat{\mathbf{r}}_i$ of object i for all $t > \tau$. The following theorem outlines the control law and states the stability result.

Theorem 2: Consider the control error (38) and suppose that Assumptions 1-4 hold. The control inputs are given by

$$\begin{aligned} v &= -k_1\cos\left(\widehat{\vartheta}_i(t)\right)\left(d_{ti} - \widehat{d}_i(t)\right) \\ w &= -k_2\left(\vartheta_{ti} - \widehat{\vartheta}_i(t)\right),\quad \forall t > \tau \qquad (39)\end{aligned}$$

where Assumption 3 specifies $\widehat{\vartheta}_i(t) \in [-\pi,\pi)$ and $k_2 \geq k_1 \geq 1$ are control gains. Assume that $\vartheta_{ti} \neq \pm\frac{\pi}{2}$. Then the error (38) asymptotically and exponentially converges to zero given any initial sensor-object configuration $\widehat{\mathbf{r}}_i(\tau) = [\widehat{d}_i(\tau)\ \widehat{\vartheta}_i(\tau)]^{\top}$.

Proof: The error $\boldsymbol{\delta}_{ti}(t)$ obeys

$$\dot{\delta}_{ti}(t) = \begin{bmatrix} -k_1 \cos^2 \widehat{\vartheta}_i(t) & 0 \\ -\frac{k_1}{\widehat{d}_i(t)} \sin \widehat{\vartheta}_i(t) \cos \widehat{\vartheta}_i(t) & -k_2 \end{bmatrix} \delta_{ti}(t) \quad (40)$$

for $t > \tau$ and where Assumption 2 claims there exists a coordinate scale such that $d_i(t) \geq 1$ in any practical scenario. We also have Assumption 3 which claims the estimated state output will belong to the adopted state space such that $\widehat{d}_i(t) \geq 1$. Note that differential equation (40) is of the form $\dot{\delta}_{ti}(t) = \mathbf{F}(\delta_{ti})\delta_{ti}(t)$ and is nonlinear since $\mathbf{F}(\delta_{ti})$ is dependent on the error. Let $\mathcal{B}(\delta_{ti}(t)) = \delta_{ti}(t)^\top \delta_{ti}(t)$ be a candidate Lyapunov function. It remains to establish that $\mathbf{F}(\delta_{ti}) + \mathbf{F}(\delta_{ti})^\top$ is negative definite. If $\widehat{\vartheta}_i(t) \neq \pm\frac{\pi}{2}$ then under the adopted assumptions it is easy to verify

$$\mathrm{tr}(\mathbf{F}(\delta_{ti}) + \mathbf{F}(\delta_{ti})^\top) < 0 \quad (2)$$

$$\det(\mathbf{F}(\delta_{ti}) + \mathbf{F}(\delta_{ti})^\top) > 0 \quad (3)$$

If $\vartheta_{ti} \neq \pm\frac{\pi}{2}$ is not a desired pose objective then clearly $\dot{\delta}_{ti}(t)$ is not at equilibrium and $w \neq 0$. Thus $\widehat{\vartheta}_i(t) \neq \pm\frac{\pi}{2}$ represent non-attractive and non-invariant manifolds in the state space. This completes the proof. ■

A relative angle $\vartheta_{ti} = \pm\pi/2 \pm \epsilon$ for any arbitrarily small $\epsilon > 0$ is stabilizable given the designed continuous controller. In practice this is quite sufficient. To achieve an exact relative angle $\vartheta_{ti} = \pm\pi/2$ requires a slight (technical) modification of the control law for v and is straightforward but results in a control function for v that is discontinuous at $\widehat{\vartheta}_i(t) = \pm\frac{\pi}{2}$. The details are omitted for brevity but are quite simple.

We thus have illustrated how a polar formulation of the problems considered can be directly exploited to yield very simple solutions in a very natural form. The controlled sensor trajectories are also very natural. We now consider an example involving a robot with unicycle kinematics (1) and initial state $\mathbf{s} = [0\ 0\ 0]^\mathrm{T}$. We have randomly placed an object (simulating a random initial sensor-object pose) in the environment. The desired relative pose is characterized solely by $\mathbf{t}_i = [2\ \ -\pi/4]^\mathrm{T}$ and $k_1 = k_2 = 0.2$. Figure 1 part (a) illustrates the sensor trajectory and part (b) illustrates the range and angle error convergence.

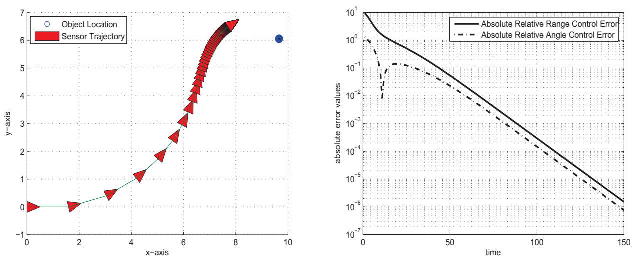

Fig. 1. *(a) shows the sensor trajectory and (b) shows the error convergence.*

From Figure 1 we note the natural and continuous sensor trajectory and the fast convergence of the errors to zero. Nothing more than Lyapunov methods were used to prove control stability. Additional examples are omitted for brevity.

V. Concluding Remarks

The problem of object localization using a mobile sensor was examined in this paper. We derived a coordinate transform and a relative sensor-object motion model that leads to a novel problem formulation where the measurements are linear in the object positions. We then apply an extended Kalman filter-like algorithm to the estimation problem. Using stochastic calculus we analyzed the convergence properties of the filter. We then illustrate that it is possible to steer the mobile sensor back to a relative sensor-object pose using a simple continuous control law. This last fact is significant since we can circumvent Brockett's negative result. The polar formulation considered in this paper provides a very natural representation of the general localization and sensor-object pose control problems. This simplifies the design of the filter and the control law (since the actual problem is represented naturally and so are the control objectives) and it also improves the performance of the estimator (as no approximate linearization of the measurements is needed).

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