SDE Minicourse Assignment

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Excercise 1 Consider the scalar Ito Stochastic Differential Equation [GMS+06]

$$dX(t) = -V'(X(t))dt + \sqrt{2\epsilon}dW(t), \tag{1}$$

with parameter $\epsilon > 0$ and a given real valued function, $V : \mathbb{R} \to \mathbb{R}$, which is a double well potential, e.g.

$$V(x) = \Delta V\,(x+1)^2(x-1)^2. \tag{2}$$

1. Why is (2) called a double well potential? Consider the case $\epsilon = 0$ in (1) and discuss, for the resulting ordinary differential equation, its fixed points and their stability.
2. When $\epsilon > 0$ there is a qualitative change in the behavior of the solution to (1): trajectories starting inside of one "well" can escape from it and swtich to the other well. The question is how long does it take for this escape to happen as a function of ϵ.

   Use the Forward Euler (Euler-Maruyama) method; can you verify numerically that

   $$\lim_{\epsilon \to 0} \epsilon\, E[\log(\tau)] = \Delta V,$$

   where τ is the exit time from the well and ΔV is the "depth of the well"?

Excercise 2

Consider the Fitzhugh-Nagumo ODE perturbed by noise, modeling for instance biological clocks, cf. [DVEM05],

$$\begin{aligned} \epsilon\, dx_t &= \left(x_t - \frac{x_t^3}{3} - y \right) dt + \sqrt{\epsilon}\, \delta_1\, dW_t^{(1)} \\ dy_t &= (x_t + a)dt + \delta_2\, dW_t^{(2)} \end{aligned} \tag{3}$$

Here $W^{(1)}$ and $W^{(2)}$ are independent scalar Wiener processes and ϵ, δ_1, δ_2 and a are non-negative constants.

1. Let $\delta_1 = \delta_2 = 0$ in the above and analyze the phase plane of the resulting ordinary differential equation. Do you see any connection with the double well potential?
2. Explore the behavior of (3) by simulating it with the Euler-Maruyama method for the following parameters: $\epsilon = 2.00e - 01,\ 1 < a < \sqrt{3}\,,\ \delta_2 = 0$. Take δ_1 such that

$$\delta_1^2 \log(\epsilon^{-1}) = \mathcal{O}(1).$$

   What do you see?
3. Can you find a related application in material science that behaves like (3)?

References

[DVEM05] R. E. Lee DeVille, Eric Vanden-Eijnden, and Cyrill B. Muratov. Two distinct mechanisms of coherence in randomly perturbed dynamical systems. *Phys. Rev. E (3)*, 72(3):031105, 10, 2005.

[GMS+06] J. Goodman, K.S. Moon, A. Szepessy, R. Tempone, and Z. Zouraris. Stochastic and Partial Differential Equations with Adapted Numerics. *Lecture Notes*, 2006.