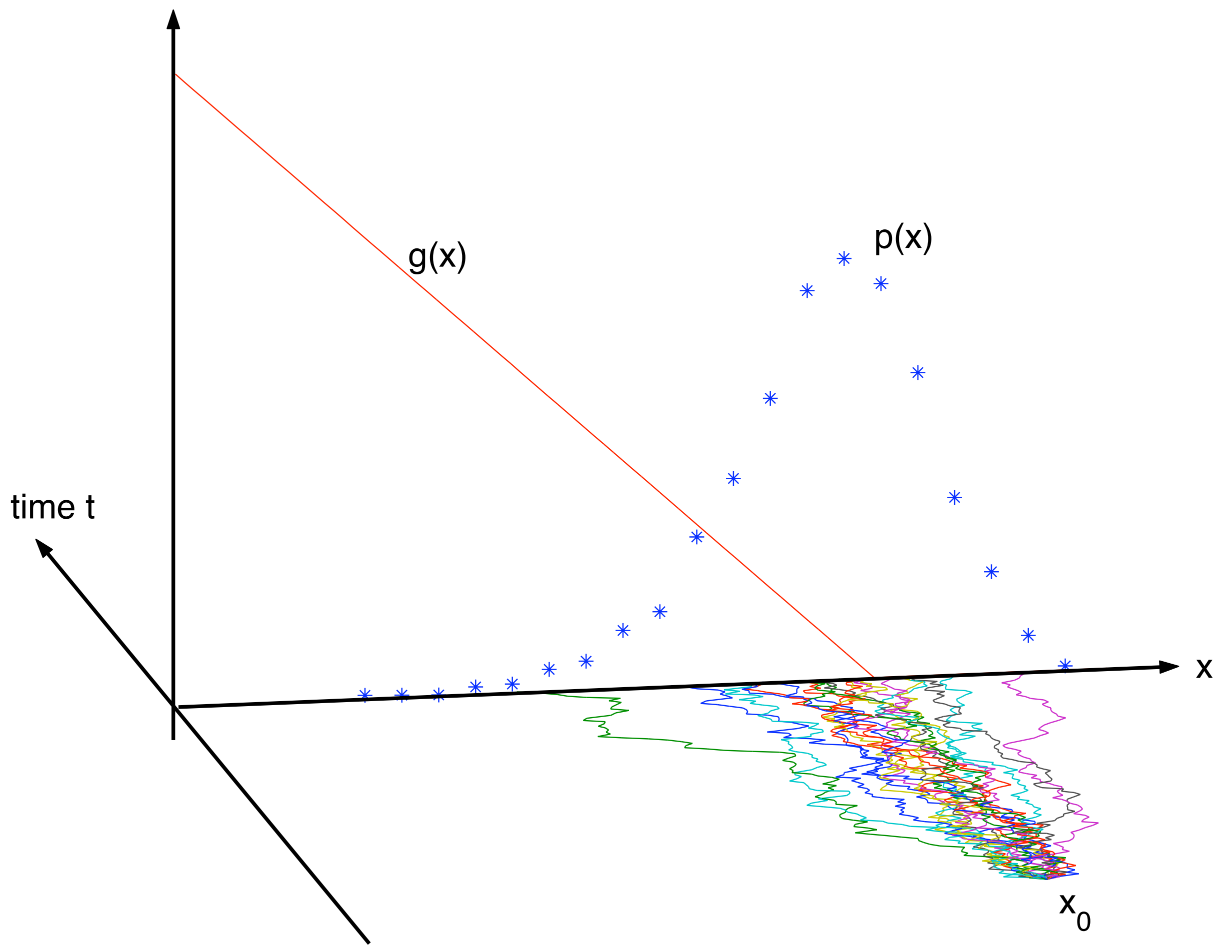
Monte Carlo Euler Weak Error Analysis

- **Problem formulation**
- **Adaptive Algorithms**
- **Error Representation**
- **Time Discretization Error Approximation**

Formulation: Weak Approximation of Diffusions

Problem: Given $g : \mathbb{R}^d \to \mathbb{R}$, approximate $E\left[g(X(T))\right]$, where $X \in \mathbb{R}^d$ solves

$$\begin{aligned} X(t) =& X(0) + \int_0^t a(s, X(s))ds \\ &+ \sum_{\ell=1}^{\ell_0} \int_0^t b^\ell(s, X(s))dW^\ell(s) \end{aligned} \tag{83}$$



Weak approximation problem.

Monte Carlo Euler Method

Let $0 = t_0 < t_1 < \ldots < t_N = T$ be given

and, for $n = 0, \ldots, N-1$, $\ell = 1, \ldots, \ell_0$

$$\begin{cases} \Delta t_n & \equiv t_{n+1} - t_n, \\ \Delta W_n^\ell & \equiv W^\ell(t_{n+1}) - W^\ell(t_n) \sim N(0, \Delta t_n), \end{cases}$$

Euler method:

Give $\overline{X}_0 = X(0)$ deterministic and define

$$\overline{X}_{n+1} = \overline{X}_n + a(t_n, \overline{X}_n)\Delta t_n + b^\ell(t_n, \overline{X}_n)\Delta W_n^\ell,$$

Monte Carlo:

Generate M realizations of $\overline{X}_N$ and approximate

$E[g(\overline{X}_N)] \approx \frac{1}{M}\sum_{j=1}^M g(\overline{X}_N(\omega_j))$

Error splitting:

$$
\begin{aligned}
E[g(X(t_N))] - \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}_N(\omega_j)) = \\
\underbrace{\Big\{E\big[g(X(T)) - g(\overline{X}_N)\big]\Big\}}_{\text{Time discretization error}} \\
+ \underbrace{\Big\{E[g(\overline{X}_N)] - \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}_N(\omega_j))\Big\}}_{\text{Statistical Error}}
\end{aligned}
$$

Observation: The choice of Δt controls the size of the time discretization error,

$$E\big[g(X(T)) - g(\overline{X}_N)\big],$$

and the choice of the number of realizations, M, controls the size of the statistical error,

$$E[g(\overline{X}_N)] - \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}_N(\omega_j)).$$

Adaptive Algorithms Given a desired accuracy, $TOL = TOL_T + TOL_S > 0$ then find

- $0 = t_0 < t_1(\omega) < \ldots < t_N = T$, a partition of [0,T]
- a number of realizations M

s.t. with probability close to one we can bound the statistical error by

$$\left| E[g(\overline{X}_N)] - \frac{1}{M}\sum_{j=1}^{M} g(\overline{X}_N(\omega_j)) \right| \leq TOL_S,$$

and the time discretization error by

$$\left| E[g(X(T)) - g(\overline{X}_N)] \right| \approx E_T \leq TOL_T,$$

while the computational work, $\sim ME[N]$, is minimized.

Monte Carlo Euler Time Discretization Error

Problem: How can we estimate the difference $E[g(X(T)) - g(\overline{X}_N)]$?

First result: An a priori estimate,

$$E[g(X(T)) - g(\overline{X}_N)] = \mathcal{O}(\Delta t_{\max})$$

We prove the result in several lemmas.

Remark 19 (Time continuous Forward Euler)
For theoretical purposes only we extend $\overline{X}$ *from* $0 = t_0 < t_1 < \ldots < t_N = T$, *to all* $t \in [0, T]$.
Let $t_n < t < t_{n+1}$, *then*

$$\begin{aligned}\overline{X}(t) = \overline{X}(t_n) &+ a(t_n, \overline{X}(t_n))(t - t_n)\\ &+ b(t_n, \overline{X}(t_n))(W(t) - W(t_n)),\end{aligned}$$

that is

$$\begin{aligned}\overline{X}(t) = \overline{X}(t_n) &+ \int_{t_n}^{t} a(t_n, \overline{X}(t_n))ds\\ &+ \int_{t_n}^{t} b(t_n, \overline{X}(t_n))dW(s)\end{aligned}$$

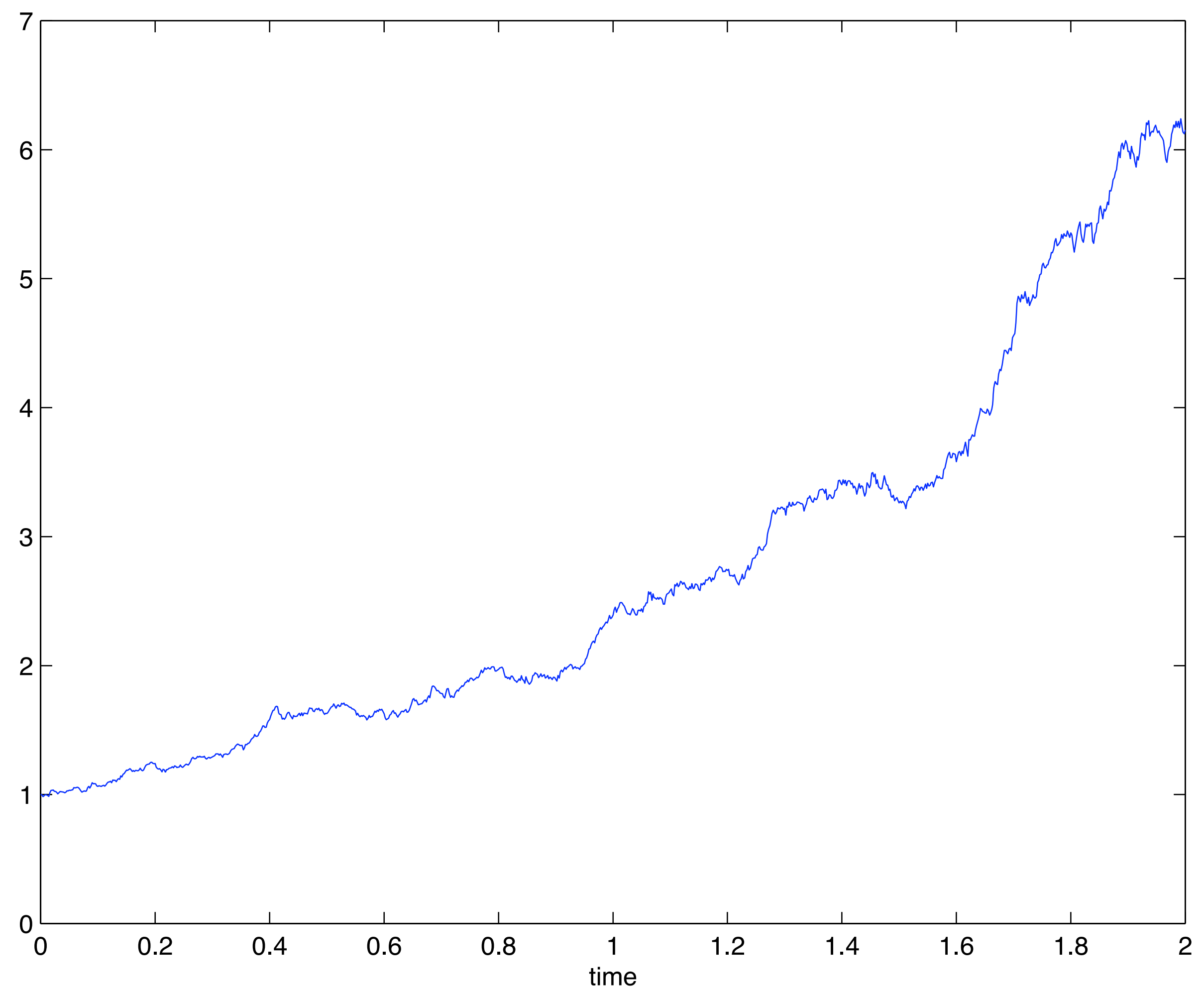
or, with the notations

$$\left.\begin{aligned}\overline{a}(s;\overline{X}) &= a(t_n,\overline{X}(t_n)),\\ \overline{b}(s;\overline{X}) &= b(t_n,\overline{X}(t_n)),\end{aligned}\right. \quad \textit{for } t_n \le s < t_{n+1}$$

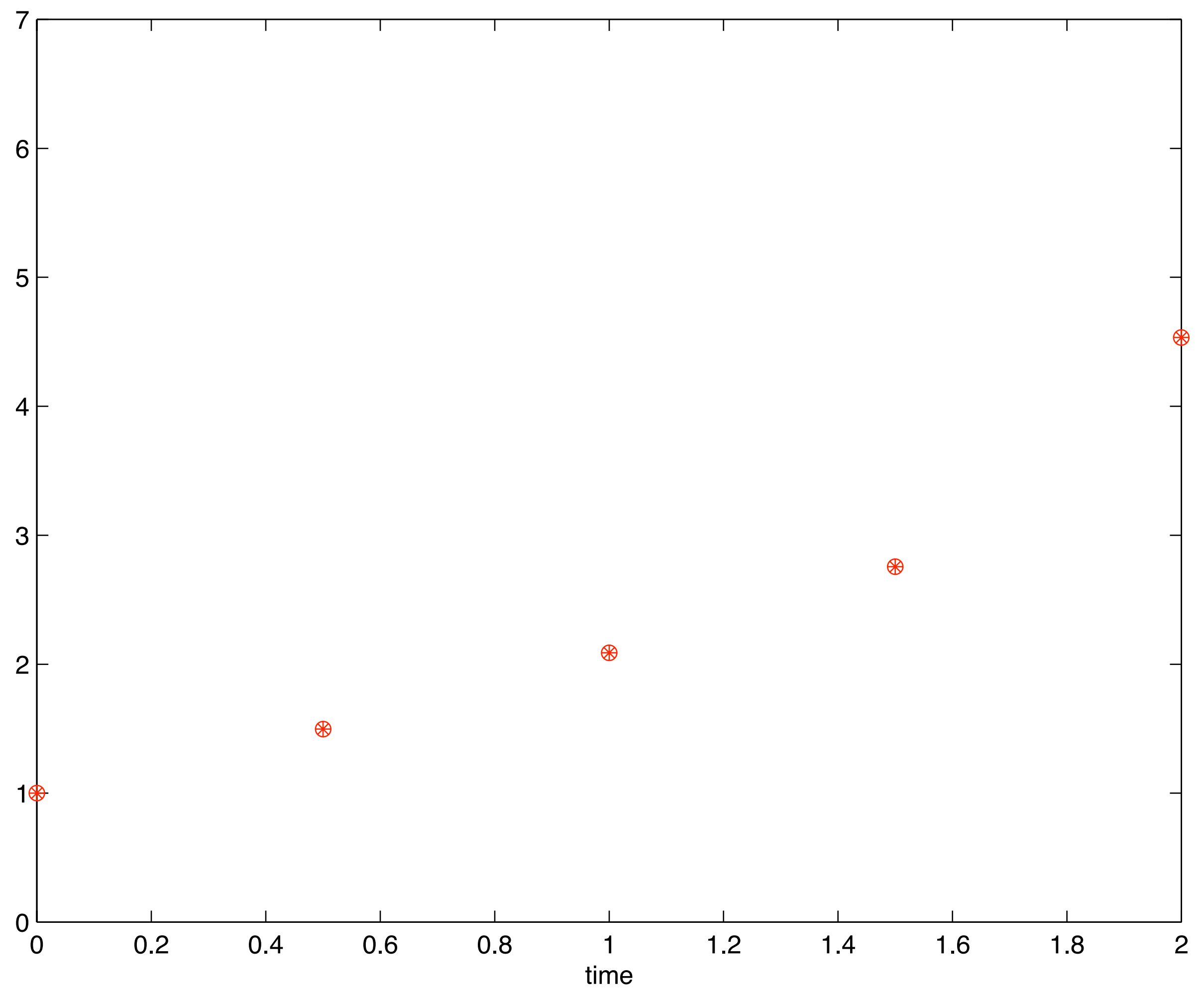
we can write

$$\begin{aligned}\overline{X}(t) = \overline{X}(t_n) &+ \int_{t_n}^{t} \overline{a}(s;\overline{X})ds\\ &+ \int_{t_n}^{t} \overline{b}(s;\overline{X})dW(s)\end{aligned}$$

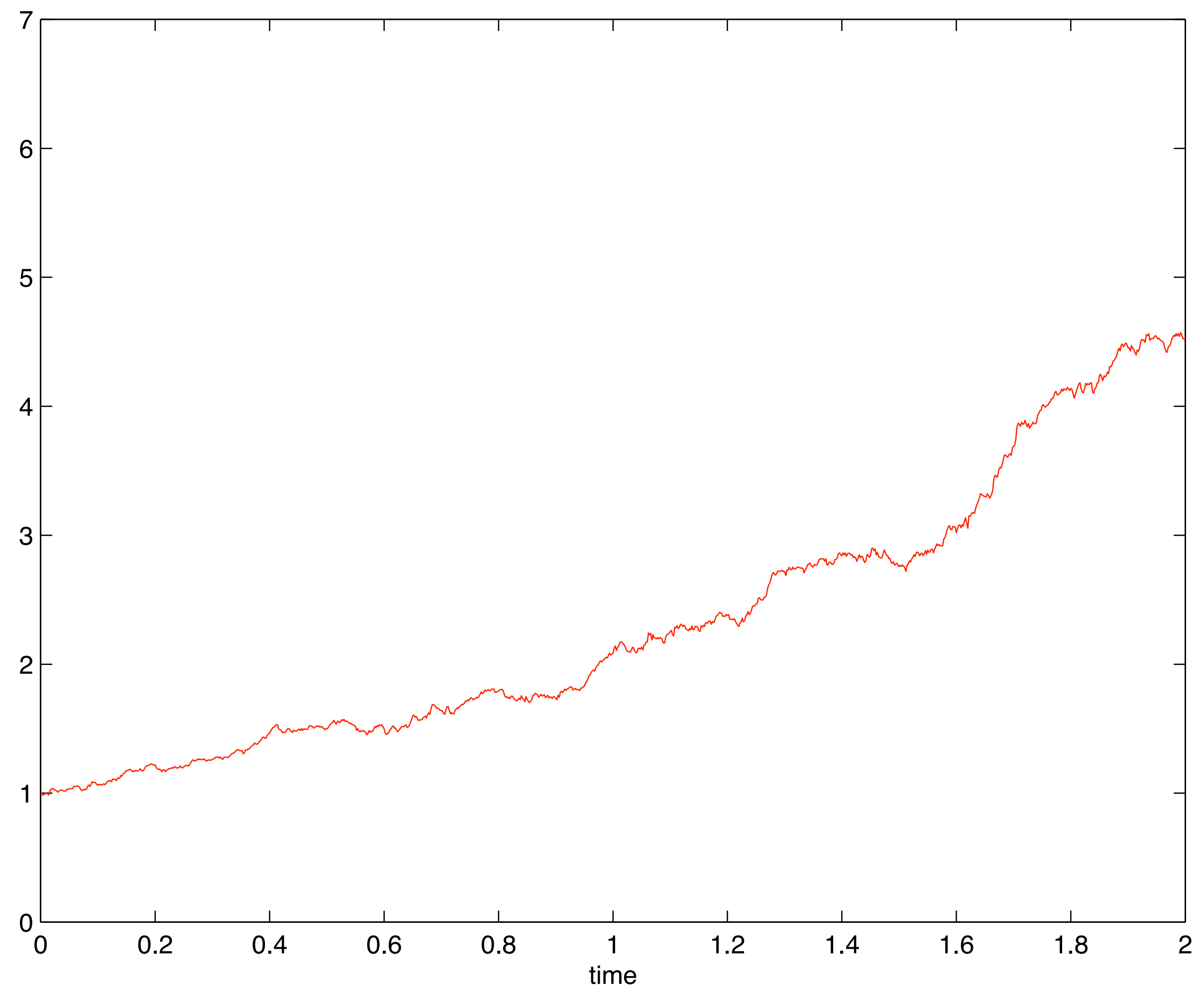
Observe that $\overline{a}$ *and* $\overline{b}$ *are simple functions and the integrals above are defined without limits.*



Exact X realization.



Discrete F. Euler approximation, $\overline{X}$.



Continuous time F. Euler approximation, $\overline{X}$.

Lemma 4 (Error representation) *Suppose that a, b and g are smooth and with appropriate decay when $|x| \to \infty$.* **Then,**

$$
\begin{aligned}
&E[g(X(T)) - g(\overline{X}(T))] = \\
&\int_0^T E\Big[\big(a_k(t,\overline{X}(t)) - \overline{a}_k(t;\overline{X})\big)\,\partial_k u(t,\overline{X}(t))\Big]dt \\
&+ \int_0^T E\Big[\big(d_{ij}(t,\overline{X}(t)) - \overline{d}_{ij}(t;\overline{X})\big)\,\partial_{ij} u(t,\overline{X}(t))\Big]dt
\end{aligned}
\tag{84}
$$

with $d_{ij} \equiv \frac{1}{2} b_i^\ell b_j^\ell$, $\overline{d}_{ij} \equiv \frac{1}{2}\overline{b}_i^\ell \overline{b}_j^\ell = (b_i^\ell b_j^\ell)(t_n, \overline{X}(t_n))$, *and the* cost to go *function*

$$
u(t,x) = E\big[g(X(T))\big|X(t) = x\big].
$$

Proof: The *cost to go* function

$$u(t,x) = E\big[g(X(T))\big|X(t) = x\big]$$

solves the Kolmogorov backward equation

$$\begin{aligned} \partial_t u + a_k \partial_k u + \tfrac{1}{2} b_k^\ell b_n^\ell \partial_{kn} u &= 0 \\ u(T,\cdot) &= g. \end{aligned}$$

Use the definition of u to represent the error

$$E[g(\overline{X}(T)) - g(X(T))] = E[u(T,\overline{X}(T)) - u(0,\overline{X}(0))]$$

Then use Ito's formula on $u(t,\overline{X}(t))$, recalling that $d\overline{X}(t) = \overline{a}(t;\overline{X}(t))dt + \overline{b}_i^\ell(t;\overline{X})dW^\ell(t)$, giving

$$\begin{aligned}
u(T,\overline{X}(T)) - u(0,\overline{X}(0))
&= \int_0^T \Big[\frac{\partial}{\partial t}u(t,\overline{X}(t)) \\
&+ \overline{a}_i(t;\overline{X})\partial_i u(t,\overline{X}(t)) \\
&+ \overline{d}_{ij}\partial_{ij}u(t,\overline{X}(t))\Big]dt \\
&+ \int_0^T \overline{b}_i^\ell(t;\overline{X})\partial_i u(t,\overline{X}(t))dW^\ell(t)
\end{aligned}$$

To conclude, use the equation satisfied by u and take expected values in the above.

Lemma 5 (Integrand Estimates) *Let*

$$t_n(t) = \max\{t_m : t_m \le t,\, m = 0 : N-1\}.$$

We have, for $t_n \le t < t_{n+1}$, $n = 0, \ldots, N-1$,

$$\begin{aligned} f_1(t) \equiv & E\big[\big(a_k(t,\overline{X}(t)) - \overline{a}_k(t;\overline{X})\big)\, \partial_k u(t,\overline{X}(t))\big] \\ = & \mathcal{O}\left(t - t_n(t)\right) = \mathcal{O}\left(\Delta t(t)\right) \end{aligned} \tag{85}$$

and

$$\begin{aligned} f_2(t) \equiv & E\big[\big(d_{ij}(t,\overline{X}(t)) - \overline{d}_{ij}(t;\overline{X})\big)\, \partial_{ij} u(t,\overline{X}(t))\big] \\ = & \mathcal{O}\left(t - t_n(t)\right) = \mathcal{O}\left(\Delta t(t)\right) \end{aligned}$$

Proof: To prove (85) recall that

$$\overline{a}(t;X) = a(t_n, \overline{X}(t_n)),$$

for $t_n \leq t < t_{n+1}$ so

$$\begin{aligned} & f_1(t_n) \\ & \quad = E\Big[\Big(a_k(t_n, \overline{X}(t_n)) - a_k(t_n, \overline{X}(t_n))\Big)\, \partial_k u(t_n, \overline{X}(t_n))\Big] \\ & \quad = 0 \end{aligned}$$

Thus, to prove (85) it is enough to show

$$|\frac{df_1}{ds}(t)| \leq C$$

for $t_n < t \leq t_{n+1}$.

Let us introduce

$$\alpha(t,x) = \big(a_k(t,\overline{X}(t)) - \overline{a}_k(t;\overline{X})\big)\,\partial_k u(t,\overline{X}(t))$$

and observe

$$f_1(t) = E[\alpha(t,\overline{X}(t)].$$

Now compute, using Ito's formula,

$$
\begin{aligned}
\frac{df_1}{dt} =& \frac{dE[\alpha(t,\overline{X}(t))]}{dt}\\
=& \lim_{h\to 0} \frac{E[\alpha(t+h,\overline{X}(t+h)) - \alpha(t,\overline{X}(t))]}{h}\\
=& \lim_{h\to 0} \frac{E[\int_t^{t+h}(\partial_t + \overline{a}_i\partial_i + \overline{d}_{ij}\partial_{ij})\alpha(s,\overline{X}(s))ds]}{h}\\
&+ \lim_{h\to 0} \frac{E[\int_t^{t+h} \overline{b}_i^\ell \partial_i \alpha(s,\overline{X}(s))dW^\ell(s)]}{h}\\
=& E[(\partial_t + \overline{a}_i\partial_i + \overline{d}_{ij}\partial_{ij})\alpha(t,\overline{X}(t))]\\
=& \mathcal{O}(1)\,.
\end{aligned}
$$

The estimate for $\frac{df_2}{dt}$ follows analogously.

Combining Lemmas 4 and 5 yields the desired a priori estimate

Theorem 22 (Forward Euler Weak error)

$$
\begin{aligned}
E[g(X(T)) - g(\overline{X}(T))] &= \int_0^T \mathcal{O}\left(t - t_n(t)\right) dt \\
&\le \mathcal{O}\left(\Delta t_{\max}\right)
\end{aligned}
$$

Remark 20 (F. Euler time discretization error) *Observe that generically we have* $\frac{df_2}{dt}(t_n) \neq 0$ *and* $\frac{df_2}{dt}(t_n) \neq 0$ *so we cannot expect to have in general that* $E[g(X(T)) - g(\overline{X}_N)] = \mathcal{O}\left(\Delta t_{\max}^{\alpha}\right)$ *with* $\alpha > 1$. *However, the order of convergence* α *may be smaller than one if* a, b *or* g *are not smooth.*

Example 12 (C. Bayer) *Consider the linear Stratonovich SDE*

$$dX_t = A_1 X \circ dW_t^1 + A_2 X \circ dW_t^2$$

with W^1, W^2 *independent Wiener processes,*

$$X(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 1 \\ -1 & -1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

and the approximation of $E[\max(r(T) - K, 0)]$, *with* $r = \sqrt{x^2 + y^2 + z^2}$, $T = 1$, $K = 1$.

Show that computations that the weak order for Forward Euler in this case is $1/2$ *instead of the usual* 1.

Adaptive Algorithm, Deterministic, Uniform Δt

1. Set $k = 0$. Give $\Delta t[0], M[0]$.
2. For each $W(\omega_j)$, $j = 1, \ldots, M[k]$ compute in the mesh $\Delta t[k]$: $\overline{X}$ and time discretization error estimates.
3. Check convergence for time error.
4. If not converged then refine $\Delta t[k]$ or increase $M[k]$. Set $k = k + 1$ and go to 2. Otherwise continue to 5.
5. For each $W(\omega_j)$, $j = 1, \ldots, M[k]$ compute in the mesh $\Delta t[k]$: only $\overline{X}$ and the estimate for the statistical error.
6. Check convergence for statistical error.
7. If not converged then increase $M[k]$, set $k = k + 1$ and go to 5. Otherwise stop, convergence achieved.

Remark 21 (Time discretization error estimation)
Use the fact that for Forward Euler with uniform time steps,

$$E[g(X(T)) - g(\bar{X}^{\Delta t}(T))] = C\Delta t + \mathcal{O}\left(\Delta t^2\right)$$

to justify an error estimate based on the difference

$$E[g(\overline{X}^{\Delta t}(T)) - g(\bar{X}^{2\Delta t}(T))]$$

Hint: Recall the use of Richardson extrapolation. How will you approximate the expected values above?