Applications to Option pricing: Black Scholes Equation

European call option: is a contract signed at time t which gives the right, but not the obligation, to buy a stock (or other financial instrument) for a fixed price K at a fixed future time $T > t$.

At time t the buyer pays the seller the amount $f(s, t; T)$ for the option contract.

What is a fair price for $f(s, t; T)$?

Assumptions

- Option price exists and is a function of time and stock value
- Continuous time trading possible
- No transaction costs
- No arbitrage in the market
- Assets can be divided when transacted
- Stocks do not pay dividends
- Stock value follows $dS/S = \mu dt + \sigma dW_t$
- Short rate is a known function of time
- Short selling (taking a negative position) is allowed

Black Scholes Equation Let $f(t, S(t))$ be the price of a European put option where $S(t)$ is the price of a stock satisfying the stochastic differential equation

$$dS = \mu S dt + \sigma S dW,$$

where the volatility σ and the drift μ are constants. Assume also the existence of a risk free paper, B, which follows

$$dB = rBdt,$$

where r, the risk free rent is a constant.

Find the partial differential equation for the "fair" price, $f(t, S(t))$.

To do so, consider a situation where we sell one of these options at time 0, receive in cash the amount

$$f(0, S(0)),$$

and then buy $\alpha(0)$ units of the stock S at its quoted price, $S(0)$. The rest of the money,

$$f(0, S(0)) - \alpha(0)S(0)$$

is invested in a risk free account with short rate r.

Now consider the time evolution of the previous portfolio,

$$I = -f + \alpha\ S + \beta B$$

for $\alpha(t), \beta(t) \in \mathbb{R}$.
Then combining the Itô formula with a self financing condition,

$$d(\alpha S + \beta B) = \alpha dS + \beta dB$$

imply

$$\begin{aligned} dI &= -df \ + \ \alpha dS \ + \ \beta dB \\ &= -(\partial_t f \ + \ \mu S \partial_s f \ + \ \frac{1}{2}\sigma^2 S^2 \partial_{ss}^2 f)dt \ - \ \partial_s f \sigma S dW \\ &\quad +\alpha(\mu S dt + \sigma S dW) + \beta r B dt \end{aligned}$$

Rearranging previous terms in dt and dW_t we obtain

$$dI = \Big(-(\partial_t f \; + \; \mu S \partial_s f \; + \; \tfrac{1}{2}\sigma^2 S^2 \partial^2_{ss} f) \; + \; (\alpha \mu S \; + \; \beta r B)\Big)\, dt \\ + \; (-\partial_s f \; + \; \alpha)\sigma S dW.$$

Now choose α such that the portfolio I becomes riskless, i.e. $\alpha = \partial_s f$, so that

$$
\begin{aligned}
dI &= \\
&\left(-(\partial_t f + \mu S \partial_s f + \frac{1}{2}\sigma^2 S^2 \partial_{ss}^2 f) + (\partial_s f \mu S + \beta r B) \right) dt \\
&= \left(-(\partial_t f + \frac{1}{2}\sigma^2 S^2 \partial_{ss}^2 f) + \beta r B \right) dt. \qquad (24)
\end{aligned}
$$

Assume also that the existence of an arbitrage opportunity is precluded, i.e. $dI = rIdt$, where r is the interest rate for riskless investments, to obtain

$$\begin{aligned} dI &= r(-f + \alpha S + \beta B)dt \\ &= r(-f + \partial_s f S + \beta B)dt. \end{aligned} \tag{25}$$

Equation (24) and (25) show that

$$\partial_t f + rsf_s + \frac{1}{2}\sigma^2 s^2 f_{ss} = rf, \quad t < T, \tag{26}$$

and finally at the maturity time T the contract value is given by definition, e.g. a standard European put option satisfies for a given exercise price K

$$f(T, s) = \max(K - s, 0).$$

The deterministic partial differential equation (26)

$$
\begin{aligned}
\partial_t f \; + \; rsf_s \; + \; \frac{1}{2}\sigma^2 s^2 f_{ss} &= rf, \quad t < T,\\
f(T,s) &= \max(K-s,0).
\end{aligned}
$$

is called the Black-Scholes equation. The existence of an adapted process $\beta(t)$ is shown in the exercise below.

Remark 13 *Please note the dependence of the option price on the stock price. The price of the option is* **relative** *to the stock price and we assume that the stock price is available at all needed times. This is a* **consistent** *pricing of the option (in terms of precluding arbitrage) wrt to the stock price.*

Exercise 8 *Replicating portfolio. It is said that the self financing portfolio, $\alpha S + \beta B$, replicates the option f. Show that there exists an adapted stochastic process $\beta(t)$, with*

$$\beta(0) = f(0, S(0)) - \partial_s f(0, S(0))S(0),$$

satisfying self financing,

$$d(\alpha S + \beta B) = \alpha dS + \beta dB,$$

with $\alpha = \partial_s f$.

Exercise 9 *Assume that $S(t)$ is the price of a single stock and that the assumptions made or the derivation of the Black-Scholes equation hold. Derive a Monte Carlo and a PDE method to determine the price of a contingent claim with with the (path dependent) payoff*

$$\int_0^T h(t, S(t))dt,$$

for a given function h.

Hint: Introduce an auxiliary variable, $Y(t)$, with $Y(0) = 0$ and $dY_t = h(t, S(t))$. Let the price of the option be $f(t, S(t), Y(t))$ and use a replicating portfolio argument.

Exercise 10 *Derive the Black-Scholes equation for a general system of stocks* $S(t) \in \mathbb{R}^d$ *of the form*

$$dS_i(t) = a_i(t, S(t))dt + \sum_{j=1}^{d} b_{ij}(t, S(t))dW_j(t), \;\; i = 1, \ldots, d$$

and the European option with final payoff $f(T, S(T)) = g(S(T))$*. Here* $g : \mathbb{R}^d \to \mathbb{R}$ *is a given function e.g.* $g(s) = \max\left(\frac{\sum_{i=1}^d s_i}{d} - K, 0\right)$.

Hint: *Generalize the one-stock derivation considering a self financing portfolio with all underlaying stocks and the option.*

Exercise 11 *Verify that the corresponding equation (26) holds if μ, σ and r are given functions of time and stock price.*

Exercise 12 *Simulation of a replicating portfolio. Assume that the previously described Black-Scholes model holds and consider the case of a bank that has written (sold) a call option on the stock S with the parameters*

$$S(0) = S_0 = 760,\ r = 0.06,\ \sigma = 0.65,\ K = S_0.$$

with an exercise date, $T = 1/4$ years. The goal of this exercise is to simulate the replication procedure described in Exercise 8, using the exact solution of the Black Scholes call price, computed by a Matlab code.

Matlab code:

```
% BS call option computation
function y = bsch(S,T,K,r,sigma);

normal = inline('(1+erf(x/sqrt(2)))/2','x');
d1 = (log(S/K)+(r+.5*sigma^2)*T)/sigma/sqrt(T);
d2 = (log(S/K)+(r-.5*sigma^2)*T)/sigma/sqrt(T);
y = S*normal(d1)-K*exp(-r*T)*normal(d2);
```

To this end, choose a number of hedging dates, N, and time steps $\Delta t \equiv T/N$. Assume that $\beta(0) = f(0, S_0) - f_S(0, S_0)S_0$ and then

- Write a code that computes the $\Delta \equiv \partial f(0, S_0)/\partial S_0$ of a call option.
- Generate a realization for $S(n\Delta t, \omega), n = 0, \ldots, N$.
- Generate the corresponding time discrete realizations for the processes α_n and β_n and the portfolio value, $I_n = \alpha_n S_n + \beta_n B_n$.
- Generate the value after settling the contract at time T,

$$\alpha_N S_N + \beta_N B_N - \max(S_N - K, 0).$$

Compute with only one realization, and several values of N, say $N = 10, 20, 40, 80$.

Questions:

What do you observe?

How would you proceed if you don't have the exact solution of the Black-Scholes equation?

Numerical Example: hedging in a B-Scholes world

$$dS/S = \mu dt + \sigma dW_t$$

Consider a call option $g(s) = \max(s - K, 0)$.

```
T = 4/12; % with 4 months life options
sigma = .47; % volatility
r = 0.04; % annual spot rate
```

Further, $K = S(0)$ (at the money options) and we only hedge for 90 days.

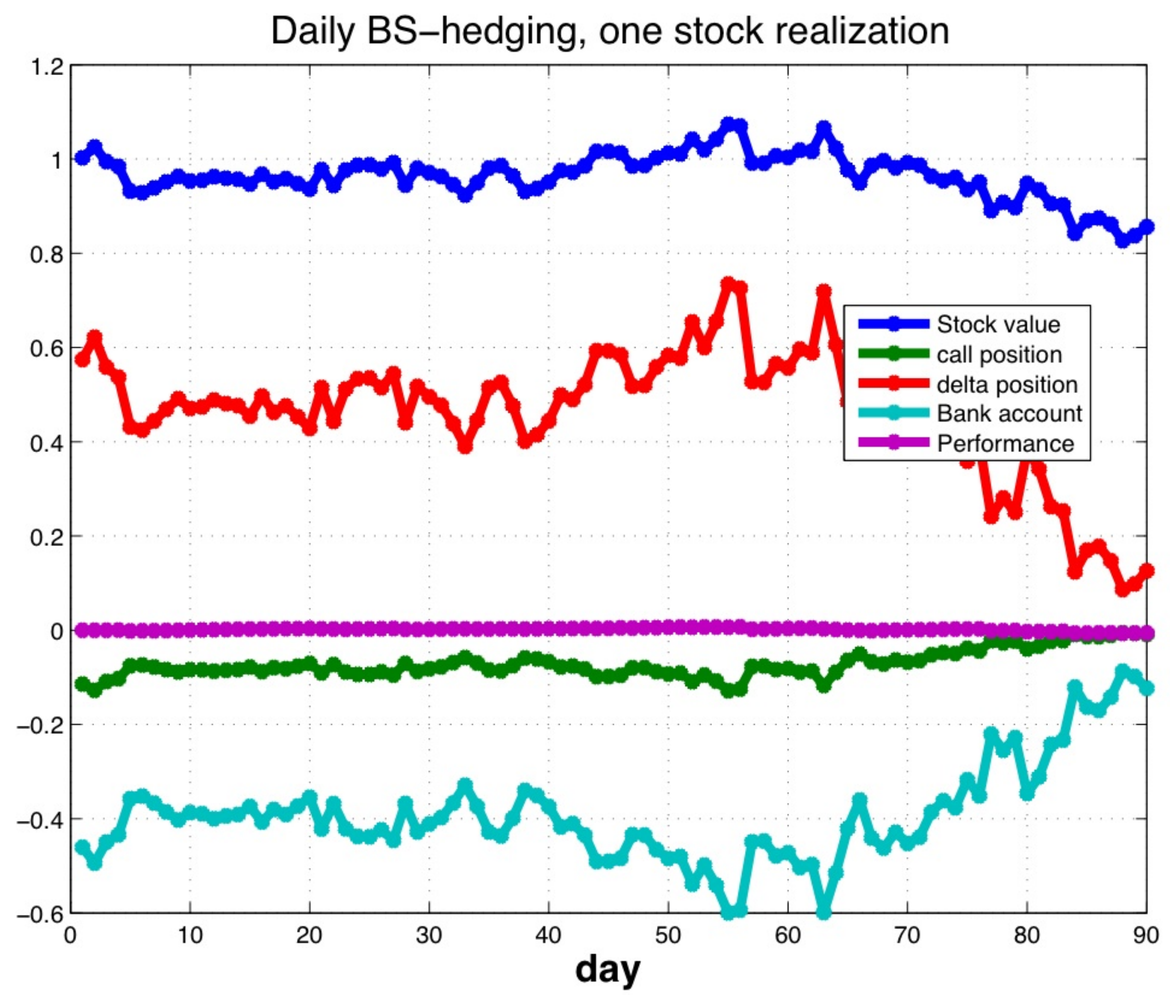
Some notations:

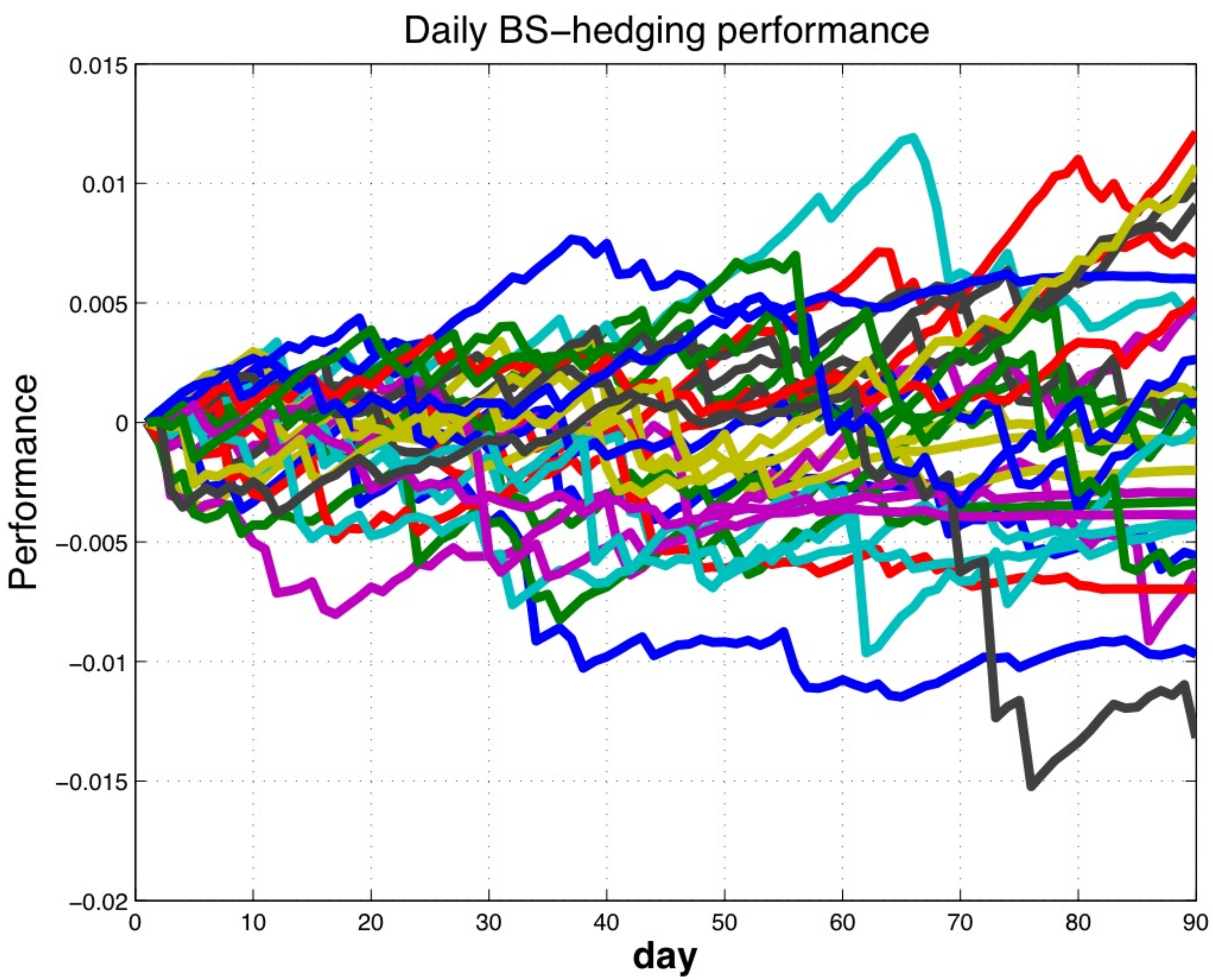
$$Delta = \Delta = \partial_s f$$

```
transac = (deltav(day)-deltav(day-1))*Sval(day);
Bankacc(day) =  Bankacc(day-1)*(1+r*dt) - transac;
Performance(day)= -callv(day)
          + deltav(day)*Sval(day)  + Bankacc(day);
```

Observe that the starting value for the bank account position is

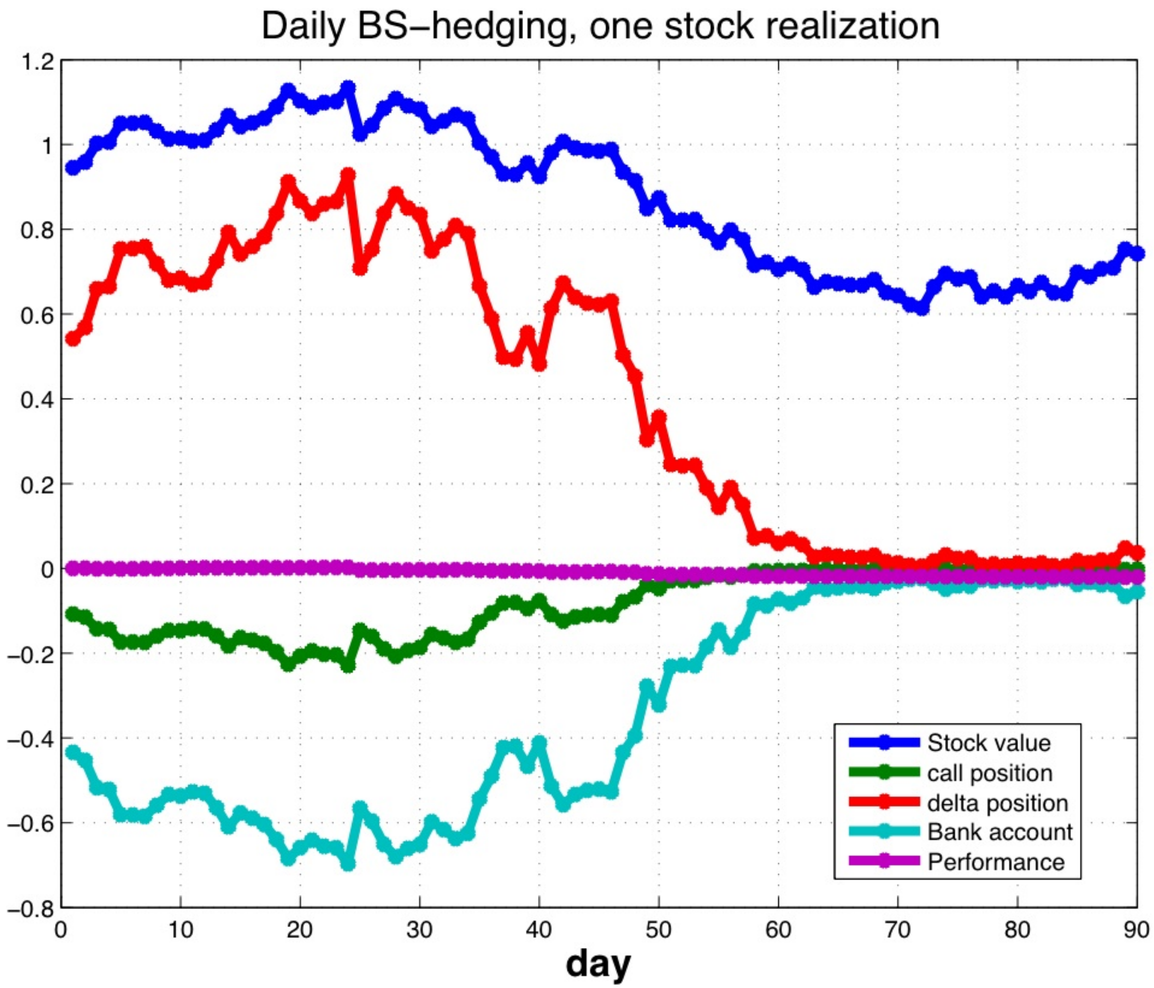
$$\beta(0) = f(0, S_0) - \Delta(0) S_0$$

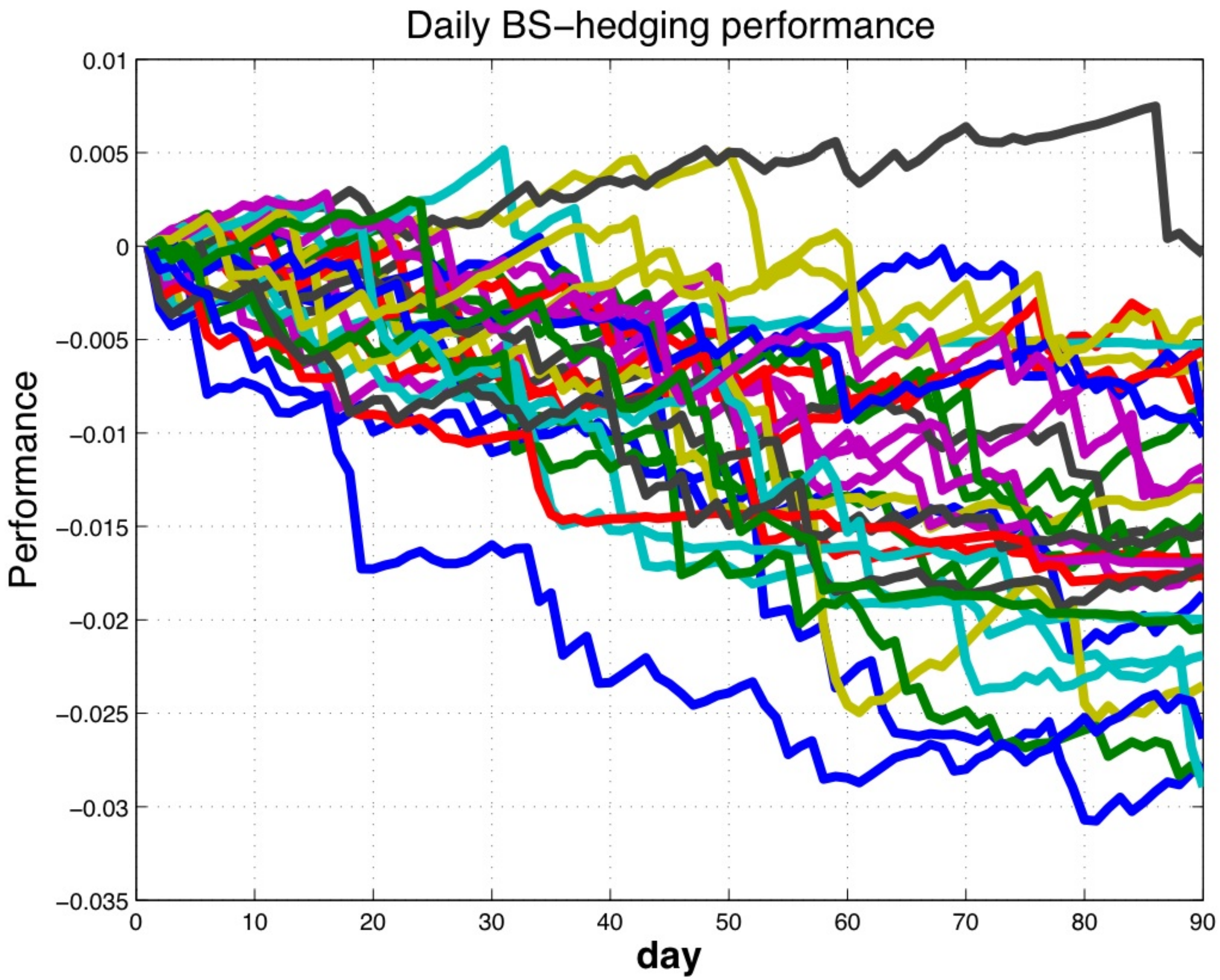

Daily BS–hedging, one stock realization
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
0
10
20
30
40
50
60
70
80
90
day
Stock value
call position
delta position
Bank account
Performance


Daily BS−hedging performance
Performance
day
0.015
0.01
0.005
0
−0.005
−0.01
−0.015
−0.02
0
10
20
30
40
50
60
70
80
90

What if we use a wrong value for σ?

Suppose that S follows B-S with 20 percent larger volatility ...


Daily BS–hedging, one stock realization
1.2
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
0
10
20
30
40
50
60
70
80
90
day
Stock value
call position
delta position
Bank account
Performance


Daily BS−hedging performance
Performance
day
0.01
0.005
0
−0.005
−0.01
−0.015
−0.02
−0.025
−0.03
−0.035
0
10
20
30
40
50
60
70
80
90