```
> restart;
with(linalg):

#Version 1: Eigenvalues based on normalized field

#First oder approximation of vector field W(x,y,z)=(uf(x,y,z),vf(x,y,z),wf(x,y,z))
uf := u + u_x*x + u_y*y + u_z*z;
vf := v + v_x*x + v_y*y + v_z*z;
wf := w + w_x*x + w_y*y + w_z*z;

#Assume that projection plane Pi is the x-y-plane at (0,0,0), i.e., W(0,0,0) points into the direction of the z-axis
u := 0;
v := 0;
assume(w>0);


#Normalized field Wn=(ufn,vfn,wfn)=W/|W|
ufn := uf/sqrt(uf^2 + vf^2 + wf^2);
vfn := vf/sqrt(uf^2 + vf^2 + wf^2);
wfn := wf/sqrt(uf^2 + vf^2 + wf^2);

#Partials of the normalized field Wn
ufn_x := diff(ufn,x):
vfn_x := diff(vfn,x):
wfn_x := diff(wfn,x):
ufn_y := diff(ufn,y):
vfn_y := diff(vfn,y):
wfn_y := diff(wfn,y):
ufn_z := diff(ufn,z):
vfn_z := diff(vfn,z):
wfn_z := diff(wfn,z):

#After computing all derivatives, we can focus on the point (0,0,0)
x := 0:
y := 0:
z := 0:


#Jacobian of Wn:
Jn := Matrix([
[factor(ufn_x), factor(ufn_y), factor(ufn_z)],
[factor(vfn_x), factor(vfn_y), factor(vfn_z)],
[factor(wfn_x), factor(wfn_y), factor(wfn_z)]
]);
```

$$uf := u + u\_x\,x + u\_y\,y + u\_z\,z$$

$$vf := v + v\_x\,x + v\_y\,y + v\_z\,z$$

$$wf := w + w\_x\,x + w\_y\,y + w\_z\,z$$

$$u := 0$$

$$v := 0$$

$$ufn := (u\_x\,x+u\_y\,y+u\_z\,z)/(u\_x^2x^2+2u\_x\,x\,u\_y\,y+2u\_x\,x\,u\_z\,z+u\_y^2y^2+2u\_y\,y\,u\_z\,z+u\_z^2z^2+v\_x^2x^2+2v\_x\,x\,v\_y\,y+2v\_x\,x\,v\_z\,z+v\_y^2y^2+2v\_y\,y\,v\_z\,z+v\_z^2z^2+w\sim^2+2w\sim w\_x\,x+2w\sim w\_y\,y+2w\sim w\_z\,z+w\_x^2x^2+2w\_x\,x\,w\_y\,y+2w\_x\,x\,w\_z\,z+w\_y^2y^2+2w\_y\,y\,w\_z\,z+w\_z^2z^2)^{\left(\frac{1}{2}\right)}$$

$$vfn := (v\_x\,x+v\_y\,y+v\_z\,z)/(u\_x^2x^2+2u\_x\,x\,u\_y\,y+2u\_x\,x\,u\_z\,z+u\_y^2y^2+2u\_y\,y\,u\_z\,z+u\_z^2z^2+v\_x^2x^2+2v\_x\,x\,v\_y\,y+2v\_x\,x\,v\_z\,z+v\_y^2y^2+2v\_y\,y\,v\_z\,z+v\_z^2z^2+w\sim^2+2w\sim w\_x\,x+2w\sim w\_y\,y+2w\sim w\_z\,z+w\_x^2x^2+2w\_x\,x\,w\_y\,y+2w\_x\,x\,w\_z\,z+w\_y^2y^2+2w\_y\,y\,w\_z\,z+w\_z^2z^2)^{\left(\frac{1}{2}\right)}$$

$$wfn := (w\sim+w\_x\,x+w\_y\,y+w\_z\,z)/(u\_x^2x^2+2u\_x\,x\,u\_y\,y+2u\_x\,x\,u\_z\,z+u\_y^2y^2+2u\_y\,y\,u\_z\,z+u\_z^2z^2+v\_x^2x^2+2v\_x\,x\,v\_y\,y+2v\_x\,x\,v\_z\,z+v\_y^2y^2+2v\_y\,y\,v\_z\,z+v\_z^2z^2+w\sim^2+2w\sim w\_x\,x+2w\sim w\_y\,y+2w\sim w\_z\,z+w\_x^2x^2+2w\_x\,x\,w\_y\,y+2w\_x\,x\,w\_z\,z+w\_y^2y^2+2w\_y\,y\,w\_z\,z+w\_z^2z^2)^{\left(\frac{1}{2}\right)}$$

$$Jn := \begin{bmatrix} \frac{u\_x}{\sqrt{w\sim^2}} & \frac{u\_y}{\sqrt{w\sim^2}} & \frac{u\_z}{\sqrt{w\sim^2}} \\ \frac{v\_x}{\sqrt{w\sim^2}} & \frac{v\_y}{\sqrt{w\sim^2}} & \frac{v\_z}{\sqrt{w\sim^2}} \\ 0 & 0 & 0 \end{bmatrix}$$

```
> #Eigenvalues of Jn
lambda := eigenvalues(Jn);
```

$$\lambda := 0, \frac{\frac{u\_x}{2}+\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2u\_x\,v\_y+v\_y^2+4v\_x\,u\_y}}{2}}{w\sim}, \frac{\frac{u\_x}{2}+\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2u\_x\,v\_y+v\_y^2+4v\_x\,u\_y}}{2}}{w\sim}$$

```
>
#Version 2: Eigenvalues based on projection

#projected M into the plane Pi (here x-y--plane)

up := u_x*x + u_y*y;
vp := v_x*x + v_y*y;
#Jacobian of projected W:
Jp := Matrix([
[factor(u_x), factor(u_y)],
[factor(v_x), factor(v_y)]
]);
```

$$up := 0$$

$$vp := 0$$

$$Jp := \begin{bmatrix} u\_x & u\_y \\ v\_x & v\_y \end{bmatrix}$$

```
> #Eigenvalues of Jp
beta := eigenvalues(Jp);
```

$$\beta := \frac{u\_x}{2}+\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2u\_x\,v\_y+v\_y^2+4v\_x\,u\_y}}{2}, \frac{u\_x}{2}+\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2u\_x\,v\_y+v\_y^2+4v\_x\,u\_y}}{2}$$

```
> #Note that beta_i = |W|*lambda_i
beta[1];
```

beta[2];

lambda[1];
lambda[2];
lambda[3];

$$\frac{u\_x}{2}+\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}$$

$$\frac{u\_x}{2}+\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}$$

$$0$$

$$\frac{\frac{u\_x}{2}+\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{w\sim}$$

$$\frac{\frac{u\_x}{2}+\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{w\sim}$$

> **#The following shows that the eigenvectors of Jn corresponding to the non-zero eigenvalues lie in the x-y-plane**

EV_JN := eigenvectors(Jn);

$$EV\_JN := \left[0,\,1,\,\left\{\left[\frac{-v\_y\,u\_z+v\_z\,u\_y}{u\_x\,v\_y-v\_x\,u\_y},\,-\frac{-v\_x\,u\_z+u\_x\,v\_z}{u\_x\,v\_y-v\_x\,u\_y},\,1\right]\right\}\right],$$

$$\left[\frac{\frac{u\_x}{2}+\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{w\sim},\,1,\,\left\{\left[1,\,-\frac{\frac{u\_x}{2}-\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{u\_y},\,0\right]\right\}\right],$$

$$\left[\frac{\frac{u\_x}{2}+\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{w\sim},\,1,\,\left\{\left[1,\,-\frac{\frac{u\_x}{2}-\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{u\_y},\,0\right]\right\}\right]$$

> **#Note how the eigenvectors of Jn corresponding to the non-zero eigenvalues lie in the x-y-plane**
#Eigenvalue; Eigenvector
EV_JN[2][1];EV_JN[2][3];
EV_JN[3][1];EV_JN[3][3];

$$\frac{\frac{u\_x}{2}+\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{w\sim}$$

$$\left\{\left[1,\,-\frac{\frac{u\_x}{2}-\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{u\_y},\,0\right]\right\}$$

$$\frac{\frac{u\_x}{2}+\frac{v\_y}{2}-\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{w\sim}$$

$$\left\{\left[\left[1, -\frac{\frac{u\_x}{2}-\frac{v\_y}{2}+\frac{\sqrt{u\_x^2-2\,u\_x\,v\_y+v\_y^2+4\,v\_x\,u\_y}}{2}}{u\_y}, 0\right]\right]\right\}$$

```
> #The following shows that the eigenvector corresponding to the 0 eigenvalue is D_W from eq. (3) in the paper.
  We did not use this in the paper, but it might be of interest.

  #define the field D_W=(u_d,v_d,w_d) from eq. (3):

  ud := factor(
  +u*v_y*w_z
  +v*w_y*u_z
  +w*u_y*v_z
  -w*v_y*u_z
  -u*w_y*v_z
  -v*u_y*w_z
  );

  vd := factor(
  +u_x*v*w_z
  +v_x*w*u_z
  +w_x*u*v_z
  -w_x*v*u_z
  -u_x*w*v_z
  -v_x*u*w_z
  );

  wd := factor(
  +u_x*v_y*w
  +v_x*w_y*u
  +w_x*u_y*v
  -w_x*v_y*u
  -u_x*w_y*v
  -v_x*u_y*w
  );


  #Note that D_W is the eigenvector of Jn corresponding to the eigenvalue 0:
  #Eigenvalue; Eigenvector
  EV_JN[1][1];EV_JN[1][3];
```

$$ud := w\sim\,(-v\_y\,u\_z+v\_z\,u\_y)$$

$$vd := -w\sim\,(-v\_x\,u\_z+u\_x\,v\_z)$$

$$wd := w\sim\,(u\_x\,v\_y-v\_x\,u\_y)$$

$$0$$

$$\left\{\left[\frac{-v\_y\,u\_z+v\_z\,u\_y}{u\_x\,v\_y-v\_x\,u\_y}, -\frac{-v\_x\,u\_z+u\_x\,v\_z}{u\_x\,v\_y-v\_x\,u\_y}, 1\right]\right\}$$

```
>
```