```
> #proof of propery 1.2 for extracting PV lines
  restart;
  with(linalg):


  # given: Taylor expansion of two vector fields
  #W1 = ( uf(x,y,z), vf(x,y,z), wf(x,y,z) )^T;
  #W2 = ( uuf(x,y,z), vvf(x,y,z), wwf(x,y,z) )^T;
  #Auxiliary Field A = (uaf, vaf, waf)^T

  uf := u + u_x*x + u_y*y + u_z*z:
  vf := v + v_x*x + v_y*y + v_z*z:
  wf := w + w_x*x + w_y*y + w_z*z:

  uuf := uu + uu_x*x + uu_y*y + uu_z*z:
  vvf := vv + vv_x*x + vv_y*y + vv_z*z:
  wwf := ww + ww_x*x + ww_y*y + ww_z*z:

  uaf := ua:
  vaf := va:
  waf := wa:


  #W1 || W2 at (0,0,0), assume transformations such that W1,W2 point
  #in the direction of z-axis:
  u := 0:
  v := 0:
  w := 1:

  uu := 0:
  vv := 0:
  ww := 1:

  #Q = W1 x W2 = (usf , vsf , wsf)^T
  usf := vf*wwf - wf*vvf:
  vsf := wf*uuf - uf*wwf:
  wsf := uf*vvf - vf*uuf:

  #partials of Q:
  usf_x := diff(usf,x):
  vsf_x := diff(vsf,x):
  wsf_x := diff(wsf,x):

  usf_y := diff(usf,y):
  vsf_y := diff(vsf,y):
  wsf_y := diff(wsf,y):

  usf_z := diff(usf,z):
  vsf_z := diff(vsf,z):
  wsf_z := diff(wsf,z):

  #F = (uff, vff, wff)^T:

  uff :=
  +usf_y*vsf_z*waf
  +vsf_y*wsf_z*uaf
  +wsf_y*usf_z*vaf
  -wsf_y*vsf_z*uaf
  -usf_y*wsf_z*vaf
  -vsf_y*usf_z*waf:
```

```
vff :=
+usf_z*vsf_x*waf
+vsf_z*wsf_x*uaf
+wsf_z*usf_x*vaf
-wsf_z*vsf_x*uaf
-usf_z*wsf_x*vaf
-vsf_z*usf_x*waf:

wff :=
+usf_x*vsf_y*waf
+vsf_x*wsf_y*uaf
+wsf_x*usf_y*vaf
-wsf_x*vsf_y*uaf
-usf_x*wsf_y*vaf
-vsf_x*usf_y*waf:

#HG := (det(Q,Q_x,A) , det(Q,Q_x,A) , det(Q,Q_z,A))^T
# = (hugf , hvgf , hwgf )^T
hugf :=
+usf*vsf_x*waf
+vsf*wsf_x*uaf
+wsf*usf_x*vaf
-wsf*vsf_x*uaf
-usf*wsf_x*vaf
-vsf*usf_x*waf:

hvgf :=
+usf*vsf_y*waf
+vsf*wsf_y*uaf
+wsf*usf_y*vaf
-wsf*vsf_y*uaf
-usf*wsf_y*vaf
-vsf*usf_y*waf:

hwgf :=
+usf*vsf_z*waf
+vsf*wsf_z*uaf
+wsf*usf_z*vaf
-wsf*vsf_z*uaf
-usf*wsf_z*vaf
-vsf*usf_z*waf:

#Correction Field G = (ugf,vgf,wgf)^T = F/|F| x HG
ugf := -(hvgf*wff - hwgf*vff) / sqrt(uff^2 + vff^2 + wff^2):
vgf := -(hwgf*uff - hugf*wff) / sqrt(uff^2 + vff^2 + wff^2):
wgf := -(hugf*vff - hvgf*uff) / sqrt(uff^2 + vff^2 + wff^2):


#partials of G
ugf_x := diff(ugf,x):
vgf_x := diff(vgf,x):
wgf_x := diff(wgf,x):

ugf_y := diff(ugf,y):
vgf_y := diff(vgf,y):
wgf_y := diff(wgf,y):

ugf_z := diff(ugf,z):
vgf_z := diff(vgf,z):
wgf_z := diff(wgf,z):

#we are interested in the behavior at (0,0,0),
```

```
#all necessary derivatives are computed
x := 0:
y := 0:
z := 0:
```

```
Warning, the protected names norm and trace have been redefined and unprotected
```

```
> #check some values
  factor(uff);
  factor(vff);
  factor(wff);
```

$$-wa\,(-v\_y\,uu\_z+v\_y\,u\_z+vv\_y\,uu\_z-u\_z\,vv\_y+v\_z\,uu\_y-uu\_y\,vv\_z-u\_y\,v\_z+u\_y$$

$$-wa\,(-v\_z\,uu\_x+v\_z\,u\_x+vv\_z\,uu\_x-u\_x\,vv\_z+v\_x\,uu\_z-uu\_z\,vv\_x-u\_z\,v\_x+u\_z$$

$$wa\,(v\_x\,uu\_y-v\_x\,u\_y-vv\_x\,uu\_y+u\_y\,vv\_x-v\_y\,uu\_x+uu\_x\,vv\_y+u\_x\,v\_y-u\_x\,v$$

```
> factor(ugf);
  factor(vgf);
  factor(wgf);
```

$$0$$

$$0$$

$$0$$

```
> #nabla G
  NG := Matrix([
  [factor(ugf_x), factor(ugf_y), factor(ugf_z)],
  [factor(vgf_x), factor(vgf_y), factor(vgf_z)],
  [factor(wgf_x), factor(wgf_y), factor(wgf_z)]
  ]):
```

```
> #absF = |F|
  absF := factor(sqrt(uff^2 + vff^2 + wff^2));
```

$$absF := (wa^2\,(4\,u\_x\,vv\_y\,uu\_x\,v\_y-2\,vv\_x\,u\_y\,uu\_x\,v\_y-2\,u\_x\,v\_y\,vv\_x\,uu\_y+2\,vv\_x$$

$$-2\,u\_x\,vv\_y\,v\_x\,uu\_y+2\,u\_x\,vv\_y\,v\_x\,u\_y+4\,vv\_z\,u\_x\,v\_z\,uu\_x+2\,vv\_z\,u\_x\,u\_z\,v\_x+$$

$$+2\,vv\_z\,u\_x\,uu\_z\,vv\_x-2\,v\_x\,u\_y\,u\_x\,v\_y+4\,vv\_x\,u\_y\,v\_x\,uu\_y-2\,u\_x\,vv\_y\,vv\_x\,u\_y$$

$$-2\,uu\_x\,vv\_y\,vv\_x\,uu\_y+2\,uu\_x\,vv\_y\,v\_x\,uu\_y+2\,u\_x\,v\_y\,v\_x\,uu\_y+2\,v\_x\,u\_y\,uu\_x\,v$$

$$+2\,v\_z\,uu\_x\,u\_z\,v\_x-2\,v\_z\,uu\_x\,u\_z\,vv\_x-2\,v\_z\,uu\_x\,uu\_z\,v\_x-2\,v\_z\,u\_x\,u\_z\,v\_x+2$$

$$+2\,v\_z\,u\_x\,uu\_z\,v\_x+2\,v\_z\,uu\_x\,uu\_z\,vv\_x+2\,vv\_z\,uu\_x\,u\_z\,vv\_x+2\,vv\_z\,uu\_x\,uu\_z\,v$$

$$+2\,vv\_x\,u\_y\,uu\_x\,vv\_y-2\,vv\_z\,u\_x\,u\_z\,vv\_x-2\,vv\_z\,u\_x\,uu\_z\,v\_x-2\,vv\_z\,uu\_x\,uu\_z\,v$$

$$-2\,uu\_x\,vv\_y\,v\_x\,u\_y+2\,vv\_x\,u\_y\,u\_x\,v\_y+2\,u\_y\,v\_z\,v\_y\,uu\_z+2\,uu\_y\,vv\_z\,v\_y\,uu\_z$$

$$-2\,u\_y\,v\_z\,vv\_y\,uu\_z-2\,vv\_z^2\,u\_x\,uu\_x-2\,vv\_x\,uu\_y^2\,v\_x-2\,u\_x^2\,vv\_y\,v\_y-2\,uu\_x$$

$$+2\,uu\_y\,v\_z\,vv\_y\,uu\_z-2\,uu\_y\,v\_z\,vv\_y\,u\_z-2\,v\_x^2\,u\_y\,uu\_y-2\,u\_z\,v\_x^2\,uu\_z-2\,v\_$$

$$-2\,u\_y\,vv\_z\,v\_y\,uu\_z-2\,uu\_z\,vv\_x^2\,u\_z+4\,uu\_y\,v\_z\,u\_y\,vv\_z+2\,u\_y\,vv\_z\,v\_y\,u\_z-2$$

$$-2\,u\_z^2\,v\_x\,vv\_x-2\,vv\_z\,u\_x^2\,v\_z-2\,vv\_z\,uu\_x^2\,v\_z-2\,u\_y\,vv\_z\,vv\_y\,u\_z-2\,uu\_y\,v$$

$$-2\,vv\_x\,u\_y^2\,v\_x-2\,uu\_y\,vv\_z\,v\_y\,u\_z-2\,u\_x\,v\_y^2\,uu\_x-2\,u\_y\,v\_z\,v\_y\,u\_z+2\,uu\_y$$

$$+2\,u\_y\,vv\_z\,vv\_y\,uu\_z-2\,uu\_y\,vv\_z\,vv\_y\,uu\_z+2\,uu\_y\,vv\_z\,vv\_y\,u\_z+4\,v\_y\,u\_z\,vv\_y$$

$$+vv\_z^2\,uu\_x^2+v\_z^2\,u\_x^2+v\_z^2\,uu\_x^2+v\_x^2\,uu\_y^2+vv\_x^2\,uu\_y^2+v\_x^2\,u\_y^2+u\_$$

$$+uu\_x^2\,vv\_y^2+uu\_z^2\,vv\_x^2+u\_z^2\,v\_x^2+vv\_x^2\,u\_y^2+u\_z^2\,vv\_x^2+uu\_z^2\,v\_x^2+v\_$$

$$+vv\_y^2\,u\_z^2+vv\_y^2\,uu\_z^2+uu\_y^2\,v\_z^2+uu\_y^2\,vv\_z^2+v\_y^2\,u\_z^2-2\,vv\_y\,uu\_z^2\,v\_$$

$$-2\,vv\_y^2\,u\_z\,uu\_z-2\,uu\_y\,vv\_z^2\,u\_y-2\,v\_y^2\,u\_z\,uu\_z-2\,u\_y^2\,v\_z\,vv\_z-2\,uu\_y\,v\_z$$

> **#compute eigenvectros and eigenvalues to prove the property**
> **eigenvectors(NG);**

$$\left[0,\,1,\,\left\{\left[-\frac{-v\_y\,uu\_z+v\_y\,u\_z+vv\_y\,uu\_z-u\_z\,vv\_y+v\_z\,uu\_y-uu\_y\,vv\_z-u\_y\,v\_z+}{v\_x\,uu\_y-v\_x\,u\_y-vv\_x\,uu\_y+u\_y\,vv\_x-v\_y\,uu\_x+uu\_x\,vv\_y+u\_x\,v\_y-}\right.\right.\right.$$

$$\left.\left.\left.-\frac{-v\_z\,uu\_x+v\_z\,u\_x+vv\_z\,uu\_x-u\_x\,vv\_z+v\_x\,uu\_z-uu\_z\,vv\_x-u\_z\,v\_x+u\_z\,vv\_}{v\_x\,uu\_y-v\_x\,u\_y-vv\_x\,uu\_y+u\_y\,vv\_x-v\_y\,uu\_x+uu\_x\,vv\_y+u\_x\,v\_y-u\_x\,vv\_}\right.\right.\right.$$

$$+4\,wa^2\,u\_x\,vv\_y\,uu\_x\,v\_y-2\,wa^2\,vv\_x^2\,u\_y\,uu\_y-2\,wa^2\,u\_x\,vv\_y\,v\_x\,uu\_y+wa^2\,v\_$$

$$-2\,wa^2\,uu\_x^2\,vv\_y\,v\_y-2\,wa^2\,vv\_y\,uu\_z^2\,v\_y+2\,wa^2\,vv\_x\,uu\_y\,uu\_x\,v\_y+2\,wa^2\,u\_$$

$$-2\,wa^2\,u\_x^2\,vv\_y\,v\_y+wa^2\,v\_y^2\,uu\_z^2+wa^2\,v\_z^2\,uu\_x^2+wa^2\,v\_z^2\,u\_x^2-2\,wa^2\,u\_$$

$$-2\,wa^2\,vv\_x\,u\_y\,uu\_x\,v\_y-2\,wa^2\,vv\_x\,uu\_y^2\,v\_x+2\,wa^2\,u\_x\,vv\_y\,v\_x\,u\_y+wa^2\,vv\_z$$

$$+2\,wa^2\,u\_x\,vv\_y\,vv\_x\,uu\_y-2\,wa^2\,vv\_z^2\,u\_x\,uu\_x-2\,wa^2\,vv\_z\,uu\_x^2\,v\_z-2\,wa^2\,vv\_$$

$$-2\,wa^2\,u\_z^2\,v\_x\,vv\_x-2\,wa^2\,uu\_z^2\,vv\_x\,v\_x-2\,wa^2\,uu\_z\,vv\_x^2\,u\_z-2\,wa^2\,v\_z^2\,u\_x$$

$$-2\,wa^2\,u\_z\,v\_x^2\,uu\_z-2\,wa^2\,u\_y\,v\_z\,vv\_y\,uu\_z+wa^2\,u\_y^2\,v\_z^2+2\,wa^2\,u\_y\,v\_z\,vv\_y$$

$$+2\,wa^2\,uu\_y\,vv\_z\,v\_y\,uu\_z+2\,wa^2\,vv\_z\,u\_x\,u\_z\,v\_x+4\,wa^2\,vv\_z\,u\_x\,v\_z\,uu\_x+2\,wa^2$$

$$-2\,wa^2\,u\_x\,vv\_y\,vv\_x\,u\_y+4\,wa^2\,vv\_x\,u\_y\,v\_x\,uu\_y+wa^2\,uu\_x^2\,v\_y^2+wa^2\,uu\_z^2\,v\_$$

$$+2\,wa^2\,u\_y\,vv\_z\,v\_y\,u\_z+wa^2\,u\_z^2\,vv\_x^2+wa^2\,vv\_x^2\,u\_y^2+wa^2\,u\_z^2\,v\_x^2+wa^2\,v$$

$$+4\,wa^2\,uu\_y\,v\_z\,u\_y\,vv\_z+wa^2\,uu\_y^2\,vv\_z^2-2\,wa^2\,v\_x^2\,u\_y\,uu\_y-2\,wa^2\,uu\_x\,vv\_y$$

$$-2\,wa^2\,u\_y\,vv\_z\,v\_y\,uu\_z-2\,wa^2\,uu\_y\,v\_z\,vv\_y\,u\_z+2\,wa^2\,uu\_y\,v\_z\,vv\_y\,uu\_z-2\,wa$$

$$-2\,wa^2\,v\_x\,uu\_y\,uu\_x\,v\_y-2\,wa^2\,uu\_y\,vv\_z\,v\_y\,u\_z+2\,wa^2\,u\_x\,v\_y\,v\_x\,uu\_y+wa^2\,u$$

$$-2\,wa^2\,u\_x\,v\_y^2\,uu\_x+2\,wa^2\,uu\_x\,vv\_y\,v\_x\,uu\_y+2\,wa^2\,v\_x\,u\_y\,uu\_x\,v\_y+wa^2\,vv\_$$

$$-2\,wa^2\,uu\_y\,v\_z\,v\_y\,uu\_z-2\,wa^2\,u\_y\,vv\_z\,vv\_y\,u\_z+wa^2\,vv\_y^2\,u\_z^2-2\,wa^2\,vv\_y^2\,u$$

$$+4\,wa^2\,uu\_z\,vv\_x\,u\_z\,v\_x-2\,wa^2\,v\_z\,uu\_x\,uu\_z\,v\_x-2\,wa^2\,v\_z\,uu\_x\,u\_z\,vv\_x+2\,wa^2$$

$$+2\,wa^2\,v\_z\,uu\_x\,uu\_z\,vv\_x+2\,wa^2\,v\_z\,u\_x\,uu\_z\,v\_x+2\,wa^2\,v\_z\,u\_x\,u\_z\,vv\_x-2\,wa^2$$

$$-2\,wa^2\,v\_z\,u\_x\,uu\_z\,vv\_x+2\,wa^2\,vv\_z\,uu\_x\,uu\_z\,v\_x+2\,wa^2\,vv\_z\,uu\_x\,u\_z\,vv\_x-2\,w$$

$$-2\,wa^2\,vv\_z\,uu\_x\,uu\_z\,vv\_x-2\,wa^2\,vv\_z\,u\_x\,uu\_z\,v\_x-2\,wa^2\,vv\_z\,u\_x\,u\_z\,vv\_x-2\,w$$

$$-2\,wa^2\,uu\_y\,v\_z^2\,u\_y+2\,wa^2\,uu\_y\,v\_z\,v\_y\,u\_z+wa^2\,u\_x^2\,v\_y^2+2\,wa^2\,vv\_x\,u\_y\,uu\_$$

$$+wa^2\,v\_x^2\,u\_y^2-2\,wa^2\,u\_y\,v\_z\,v\_y\,u\_z-2\,wa^2\,vv\_x\,u\_y^2\,v\_x+2\,wa^2\,vv\_x\,u\_y\,u\_x\,v$$

$$+4\,wa^2\,v\_y\,u\_z\,vv\_y\,uu\_z-2\,wa^2\,uu\_x\,vv\_y\,v\_x\,u\_y+2\,wa^2\,uu\_y\,vv\_z\,vv\_y\,u\_z-2\,wa$$

$$-2\,wa^2\,uu\_y\,vv\_z\,vv\_y\,uu\_z-2\,wa^2\,u\_y^2\,v\_z\,vv\_z+2\,wa^2\,u\_y\,vv\_z\,vv\_y\,uu\_z\Big)^{\left(\frac{1}{2}\right)},\,2,$$

$$\left[0,\,\frac{v\_x\,uu\_y-v\_x\,u\_y-vv\_x\,uu\_y+u\_y\,vv\_x-v\_y\,uu\_x+uu\_x\,vv\_y+u\_x\,v\_y-u\_x\,v}{-v\_z\,uu\_x+v\_z\,u\_x+vv\_z\,uu\_x-u\_x\,vv\_z+v\_x\,uu\_z-uu\_z\,vv\_x-u\_z\,v\_x+u\_z\,v}\right.$$

$$\left[1, \frac{uu\_y\, vv\_z - u\_y\, vv\_z + u\_z\, vv\_y - vv\_y\, uu\_z + v\_y\, uu\_z + u\_y\, v\_z - v\_z\, uu\_y - v\_y\, u}{-v\_z\, uu\_x + v\_z\, u\_x + vv\_z\, uu\_x - u\_x\, vv\_z + v\_x\, uu\_z - uu\_z\, vv\_x - u\_z\, v\_x + u\_z\, v}\right]$$

>