Computation tree logics CTL and CTL^*.

Model-checking CTL and CTL^*.

Logics of programs PDL and $\mathcal{L}_\mu$.

Embedding CTL and PDL into $\mathcal{L}_\mu$

Priorean and Ockhamist temporal logics

William van Ockham - the meaning of a temporal formula is about a particular sequence of events

Arthur Prior - temporal reasoning about a state and the related future (and past) states.

LTL is Ockhamist.

Tense Logic: A simple temporal logic of the Priorean type:

$\varphi ::= \bot \mid p \mid \varphi \Rightarrow \varphi \mid \Diamond\varphi \mid \Diamond^{-1}\varphi$

Semantics: Bimodal frames of the form $\langle W, R, R^{-1}\rangle$.

Axiomatisation: $\mathbf{K}_{\Diamond}$, $\mathbf{K}_{\Diamond^{-1}}$, $N_{\Diamond}$, $N_{\Diamond^{-1}}$ and

$$\varphi \Rightarrow \Box\Diamond^{-1}\varphi \text{ and } \varphi \Rightarrow \Box^{-1}\Diamond\varphi$$

$\Diamond\varphi$ - φ is possible in the next time instant

$\Diamond p \wedge \Diamond\neg p$ is satisfiable in tense logic, whereas $\circ p \wedge \circ\neg p$ is not in LTL.

Computation Tree Logic CTL^*

Syntax - state formulas φ and path formulas ψ:

$$\varphi ::= \bot \mid p \mid \exists\psi \qquad \psi ::= \varphi \mid \psi \Rightarrow \psi \mid \circ\psi \mid (\psi \mathsf{U} \psi)$$

Models: $M = \langle W, R, V\rangle$, R - serial,
$\mathsf{path}_M(s) = \{s_0 s_1 \ldots \in W^\omega : s_0 = s, s_i R s_{i+1}, i < \omega\}$

$M, s \not\models \bot$		
$M, s \models p$	iff	$p \in V(s)$
$M, s \models \exists\psi$	iff	there exists a $p \in \mathsf{path}_M(s)$ such that $M, p \models \psi$
$M, p \models \varphi$	iff	$M, p_0 \models \varphi$ for state formulas φ
$M, p \models \psi_1 \Rightarrow \psi_2$	iff	either $M, p \not\models \psi_1$ or $M, p \models \psi_2$
$M, p \models \circ\psi$	iff	$M, p^{(1)} \models \psi$
$M, p \models (\psi_1 \mathsf{U} \psi_2)$	iff	there exists a $k < \omega$ such that
		$M, p^{(i)} \models \psi_1$ for $i = 0, \ldots, k-1$, and $M, p^{(k)} \models \psi_2$

Abbreviations. Exercises on $\models$ in CTL^*

The LTL abbreviations related to $\circ$ and $(.\mathrm{U}.)$ and

$$\forall\psi \rightleftharpoons \neg\exists\neg\psi.$$

Unlike LTL, where we write $p, i \models \psi$, we write $M, p^{(i)}\psi$ in CTL.

Exercise 1 Show that, given an LTL formula ψ, $\langle W, R, V, \{s\}\rangle$ contains a satisfying behaviour for φ iff $\langle W, R, V\rangle, s \models_{CTL^*} \exists\varphi$.

Exercise 2 Rewrite the definition of $\models$ in CTL^* in the form $M, p, i \models \varphi$, where i is a position in the path p, like in LTL for both state and path φ. To this end use $\mathrm{path}_M(p, i)$ to denote the set of all R-paths q in M such that $q_0 \ldots q_i = p_0 \ldots p_i$.

Exercise 3 Show that if $\varphi_1 \Leftrightarrow \varphi_2$is valid, then $[\varphi_1/p]\chi \Leftrightarrow [\varphi_2/p]\chi$ is valid too for state formulas φ_1, φ_2 and arbitrary χ. Why is this not guaranteed to hold about path φ_1, φ_2?

Computation tree logic CTL (without the *)

Syntax: $\varphi ::= \bot \mid p \mid \varphi \Rightarrow \varphi \mid \exists \circ \varphi \mid \exists(\varphi \mathsf{U} \varphi) \mid \forall(\varphi \mathsf{U} \varphi)$.

Semantics: $\bot$, $p \in \mathbf{L}$ and $\Rightarrow$ - as usual;

$M, s \models \exists \circ \varphi$	iff	there exists an $s' \in R(s)$ such that $M, s' \models \varphi$;
$M, s \models \exists(\varphi \mathsf{U} \psi)$	iff	there exists an $n < \omega$ and an $s_0 s_1 \ldots \in \mathrm{path}_M(s)$ s. t.
		$M, s_i \models \varphi,\ i = 0, \ldots, n-1,$ and $M, s_n \models \psi$;
$M, s \models \forall(\varphi \mathsf{U} \psi)$	iff	for every $s_0 s_1 \ldots \in \mathrm{path}_M(s)$
		there exists an $n < \omega$ such that
		$M, s_i \models \varphi,\ i = 0, \ldots, n-1,$ and $M, s_n \models \psi$.

Abbreviations:

$$\forall \circ \varphi \rightleftharpoons \neg \exists \circ \neg \varphi$$

$$\exists \Diamond \varphi \rightleftharpoons \exists(\top \mathsf{U} \varphi),\ \forall \Diamond \varphi \rightleftharpoons \forall(\top \mathsf{U} \varphi),\ \exists \Box \varphi \rightleftharpoons \neg \forall \Diamond \neg \varphi,\ \forall \Box \varphi \rightleftharpoons \neg \exists \Diamond \neg \varphi.$$

Note that $\forall \circ \varphi$ is an abbreviation, but $\forall(\varphi \mathsf{U} \psi)$ is basic in CTL.

Expressive power of CTL versus LTL

An LTL formula defines a class (linear) behaviours (in Kripke model M.)

There is no means in LTL to express whether M contains other behaviours that are related to a satisfying one.

$M, p, i \models \exists\psi$ means that M has a behaviour like p until time i that satisfies ψ from time i onwards.

E.g. $\Box(p \wedge \exists \circ \neg p)$ is satisfiable in CTL.

For LTL φ, if $\mathrm{Var}(\varphi) \subseteq \mathbf{L}$ then

$$M_{\mathbf{L}} = \langle \mathcal{P}(\mathbf{L}), \mathcal{P}(\mathbf{L}) \times \mathcal{P}(\mathbf{L}), \mathcal{P}(\mathbf{L}), \lambda s.s \rangle$$

satisfies φ. Unless φ is valid,

$$M_{\mathbf{L}}, s \not\models \forall \circ \neg\varphi.$$

Expressive power of CTL versus LTL continued

Definition 1 Given a CTL^* formula φ, φ^d stands for the LTL formula obtained by erasing all the occurrences of $\exists$ in φ.

Proposition 1 A CTL^* formula φ is equivalent to a formula of the form $\forall\psi$ in which ψ is an LTL iff it is equivalent to $\forall\varphi^d$.

Definition 2 Given $M = \langle W, R, V\rangle$ and an R-path p in M,
$M_p = \langle W_p, R_p, V_p\rangle$ where

$$W_p = \{s_i : i < \omega\}$$
$$R_p = R \cap W_p^2 = \{\langle s_i, s_{i+1}\rangle : i < \omega\}$$
$$V_p = V|_{W_p}.$$

Fact 1 Let $\langle W, R\rangle$ be a tree. Then $M_p, p \models \varphi^d$ iff $M_p, s_0 \models \varphi$ where s_0 is the first state of p.

Expressive power of CTL versus LTL continued

Proof: We can assume that $\langle W, R\rangle$ is a tree in the considered $M = \langle W, R, V\rangle$.

Let $s_0 \in W$ and $p = s_0 s_1 \ldots \in \text{path}_m(s_0)$

Let φ be equivalent to $\forall\psi$, where ψ is in LTL. Let $s_0 \in W$. Then

$$
\begin{array}{lll}
M, s_0 \models \varphi & \text{iff} & \text{for all paths } p \in \text{path}_M(s_0) \quad M, p \models \psi \\
& \text{iff} & \text{for all paths } p \in \text{path}_M(s_0) \quad M_p, p \models \psi \\
& \text{iff} & \text{for all paths } p \in \text{path}_M(s_0) \quad M_p, s_0 \models \varphi \\
& \text{iff} & \text{for all paths } p \in \text{path}_M(s_0) \quad M_p, p \models \varphi^d \\
& \text{iff} & \text{for all paths } p \in \text{path}_M(s_0) \quad M, p \models \varphi^d \\
& \text{iff} & \text{for all paths } p \in \text{path}_M(s_0) \quad M, s_0 \models \forall\varphi^d.
\end{array}
$$

$\dashv$

Expressive power of CTL versus LTL continued

Proposition 2 The LTL formula $\exists\Box\Diamond p$ has no equivalent in CTL.

Model-checking CTL formulas

The efficiency of model-checking is the main benefit of using CTL. Unlike LTL, we can define

$$[\![\varphi]\!]_M = \{w \in W : M, w \models \varphi\}.$$

Model-checking CTL amounts to calculating $[\![\varphi]\!]_M$:

$$\begin{array}{lcl}
[\![\bot]\!]_M & = & \emptyset \\
[\![p]\!]_M & = & V(p) \\
[\![\varphi \Rightarrow \psi]\!]_M & = & [\![\psi]\!]_M \cup W \setminus [\![\varphi]\!]_M \\
[\![\exists \circ \varphi]\!]_M & = & \{w \in W : R(w) \cap [\![\varphi]\!]_M \neq \emptyset\} = R^{-1}([\![\varphi]\!]_M)
\end{array}$$

Model-checking CTL: $[\![\exists(\psi \mathsf{U} \chi)]\!]_M$ and $[\![\forall(\psi \mathsf{U} \chi)]\!]_M$

$$\models_{CTL} \exists(\psi \mathsf{U} \chi) \Leftrightarrow \chi \vee (\psi \wedge \exists \circ \exists(\psi \mathsf{U} \chi))$$

and

$$\models_{CTL} \forall(\psi \mathsf{U} \chi) \Leftrightarrow \chi \vee (\psi \wedge \forall \circ \forall(\psi \mathsf{U} \chi)).$$

entail

$$[\![\exists(\psi \mathsf{U} \chi)]\!]_M = [\![\chi]\!]_M \cup ([\![\psi]\!]_M \cap R^{-1}([\![\exists(\psi \mathsf{U} \chi)]\!]_M))$$

and

$$[\![\forall(\psi \mathsf{U} \chi)]\!]_M = [\![\chi]\!]_M \cup ([\![\psi]\!]_M \cap W \setminus R^{-1}(W \setminus [\![\forall(\psi \mathsf{U} \chi)]\!]_M)).$$

Let $F^{\exists}_{\psi,\chi}, F^{\forall}_{\psi,\chi} : \mathcal{P}(W) \to \mathcal{P}(W)$ and

$$F^{\exists}_{\psi,\chi}(X) = [\![\chi]\!]_M \cup ([\![\psi]\!]_M \cap R^{-1}(X))$$
$$F^{\forall}_{\psi,\chi}(X) = [\![\chi]\!]_M \cup ([\![\psi]\!]_M \cap W \setminus R^{-1}(W \setminus X))$$

$[\![\exists(\psi \mathsf{U} \chi)]\!]_M$ **and** $[\![\forall(\psi \mathsf{U} \chi)]\!]_M$ **continued**

$$F^{\exists}_{\psi,\chi}(X) = [\![\chi]\!]_M \cup ([\![\psi]\!]_M \cap R^{-1}(X))$$
$$F^{\forall}_{\psi,\chi}(X) = [\![\chi]\!]_M \cup ([\![\psi]\!]_M \cap W \setminus R^{-1}(W \setminus X))$$

Fact 2 $F^{\exists}_{\psi,\chi}$ and $F^{\forall}_{\psi,\chi}$ are monotonic:

$$X, Y \subseteq W \text{ and } X \subseteq Y \text{ imply } F^{*}_{\psi,\chi}(X) \subseteq F^{*}_{\psi,\chi}(Y),\ * \in \{\exists, \forall\}.$$

Proposition 3 Let $F : \mathcal{P}(W) \to \mathcal{P}(W)$ be monotonic. Let $W' \subseteq W$ be the least subset of W such that $F(W') \subseteq W'$, i.e., let W' be the least pre-fixpoint of F. Then W' is also the least fixpoint of F, i.e. the least subset of W such that $F(W') = W$.

Proof: $F(W') \subseteq W'$ implies $F(F(W')) \subseteq F(W')$, whence $F(W')$ is a pre-fixpoint of F too, and therefore $W' \subseteq F(W')$. $\dashv$

$[\![\exists(\psi \mathsf{U} \chi)]\!]_M$ and $[\![\forall(\psi \mathsf{U} \chi)]\!]_M$ continued

Proposition 4 Let p, φ, ψ be in the vocabulary of M and

$$[\![\chi \vee (\psi \wedge \exists \circ p)]\!]_M \subseteq [\![p]\!]_M, \text{ resp. } [\![\chi \vee (\psi \wedge \forall \circ p)]\!]_M \subseteq [\![p]\!]_M.$$

Then $[\![\exists(\psi \mathsf{U} \chi)]\!]_M \subseteq [\![p]\!]_M$, resp., $[\![\forall(\psi \mathsf{U} \chi)]\!]_M \subseteq [\![p]\!]_M$.

Proof: $[\![\exists(\psi \mathsf{U} \chi)]\!]_M \subseteq [\![p]\!]_M$ can be proved by induction on n in

$s \in [\![p]\!]_M$, provided that there is a $s_0 s_1 \ldots \in \mathsf{path}_M(s)$ such that there is a $k \leq n$ such that $M, s_k \models \chi$ and $M, s_i \models \psi$ for $i = 0, \ldots, k-1$.

The second inclusion requires transfinite induction to prove. $\dashv$

Corollary 1 $[\![\exists(\psi \mathsf{U} \chi)]\!]_M$ and $[\![\forall(\psi \mathsf{U} \chi)]\!]_M$ are the least fixed points of $F^{\exists}_{\psi,\chi}$ and $F^{\forall}_{\psi,\chi}$, respectively.

$[\![\exists(\psi \mathsf{U} \chi)]\!]_M$ and $[\![\forall(\psi \mathsf{U} \chi)]\!]_M$ continued

$F^k(X) \rightleftharpoons \underbrace{F(\ldots F(}_{k \text{ times}} X)\ldots)$.

Proposition 5 Let $|W| < \omega$ and $F : \mathcal{P}(W) \to \mathcal{P}(W)$ be monotonic. Then $F^{k+1}(\emptyset) = F^k(\emptyset)$ for some $k \leq |W|$ and $F^k(\emptyset)$ is the least solution of $X = F(X)$ wrt $\subset$ in $\mathcal{P}(W)$.

Model-checking CTL^*

CTL^* model-checking is about calculating $[\![\varphi]\!]_M$ for CTL^* state formulas φ.

Technique: interleaved stages of LTL and CTL model-checking.

To calculate $[\![\exists\psi]\!]_M$:

1. Find an LTL formula θ and state formulas $\chi_1, \ldots, \chi_k$ such that

$$\psi \doteq [\chi_1/q_1, \ldots, \chi_k/q_k]\theta.$$

2. Extend $M = \langle W, R, V \rangle$ to $M' = \langle W, R, V' \rangle$ for $\mathbf{L} \cup \{q_1, \ldots, q_k\}$ by putting

$$V'(q_i) = [\![\chi_i]\!]_M, \; i = 1, \ldots, k.$$

Then

$$[\![\exists\psi]\!]_M = [\![\exists\theta]\!]_{M'}.$$

3. Use LTL model-checking to calculate $[\![\exists\theta]\!]_{M'}$.

Propositional Dynamic Logic (PDL)

Syntax: propositional variables p, q, ..., and program variables a, b, ...; formulas φ and program terms α:

$$\alpha \quad ::= \quad Id \mid a \mid \alpha \cup \alpha \mid \alpha ; \alpha \mid \alpha^* \mid \varphi ?$$

$$\varphi \quad ::= \quad \bot \mid p \mid \varphi \Rightarrow \varphi \mid \langle \alpha \rangle \varphi$$

Semantics: $M = \langle W, R, V \rangle$; $R(a) \subset W \times W$ for program variables a.

$$[\![\bot]\!]_M = \emptyset, \qquad [\![p]\!]_M = V(p),$$

$$[\![\varphi \Rightarrow \psi]\!]_M = [\![\psi]\!]_M \cup W \setminus [\![\varphi]\!]_M, \qquad [\![\langle \alpha \rangle \varphi]\!]_M = ([\![\alpha]\!]_M)^{-1}([\![\varphi]\!]_M);$$

$$[\![Id]\!]_M = Id_W = \{\langle w, w \rangle : w \in W\}, \qquad [\![a]\!]_M = R(a),$$

$$[\![\alpha \cup \beta]\!]_M = [\![\alpha]\!]_M \cup [\![\beta]\!]_M, \qquad [\![\alpha ; \beta]\!]_M = [\![\alpha]\!]_M \circ [\![\beta]\!]_M,$$

$$[\![\alpha^*]\!]_M = ([\![\alpha]\!]_M)^*, \qquad [\![\varphi ?]\!]_M = ([\![\varphi]\!]_M)^2 \cap Id_W.$$

$\circ$ - relation composition; $([\![\alpha]\!]_M)^*$ - the reflexive and transitive closure of $[\![\alpha]\!]_M$.

Definition of PDL continued

$[\alpha]\varphi \rightleftharpoons \neg\langle\alpha\rangle\neg\varphi$

Exercise 4 Prove that the following formulas are valid in PDL:

$$\langle Id\rangle\varphi \Leftrightarrow \varphi,\ \langle\varphi?\rangle\psi \Leftrightarrow \varphi\wedge\psi,\ \langle\alpha\cup\beta\rangle\varphi \Leftrightarrow \langle\alpha\rangle\varphi\vee\langle\beta\rangle\varphi,\ \langle\alpha;\beta\rangle\varphi \Leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi.$$

Corollary 2 $*$-free PDL can be viewed as multimodal $\mathbf{K}$.

A Hoare triple $\{P\}\mathbf{code}\{Q\}$ can be written in PDL as $P \Rightarrow [[\![\mathbf{code}]\!]]Q$.

The semantics of a **while**-programming language written in PDL:

$$[\![\mathbf{skip}]\!] \rightleftharpoons Id$$

$$[\![v := b]\!] \rightleftharpoons a_{v:=b}, \qquad b \Leftrightarrow [a_{v:=b}]v,\ v' \Leftrightarrow [a_{v:=b}]v' \text{ for variables } v' \not\equiv v$$

$$[\![\mathbf{if}\ b\ \mathbf{then}\ S_1\ \mathbf{else}\ S_2]\!] \rightleftharpoons (b?;[\![S_1]\!]) \cup ((\neg b)?;[\![S_2]\!])$$

$$[\![\mathbf{while}\ b\ \mathbf{do}\ S]\!] \rightleftharpoons (b? : [\![S]\!])^*;(\neg b)?$$

More facts about PDL

PDL has the finite model property and a complete axiomatisaton. Intersection $\cap$ can be allowed in program terms:

$$[\![\alpha \cap \beta]\!]_M = [\![\alpha]\!]_M \cap [\![\beta]\!]_M,$$

Decidability still holds in that extension, but the axiomatisation of $PDL^{\cap}$ is rather difficult.

Book: Harel, D., Kozen, D., and J. Tiuryn, *Dynamic Logic*, Cambridge, MA, MIT Press, 2000.

The modal μ-calculus $\mathcal{L}_\mu$

In CTL model checking we used that $[\![\exists(\varphi \mathsf{U} \psi)]\!]_M$ and $[\![\forall(\varphi \mathsf{U} \psi)]\!]_M$ are lfps of

$$F^{\exists}_{\varphi,\psi} = \lambda X.[\![\chi \vee (\psi \wedge \exists \circ p)]\!]_{M^X_p} \text{ and } F^{\forall}_{\varphi,\psi} = \lambda X.[\![\chi \vee (\psi \wedge \forall \circ p)]\!]_{M^X_p}.$$

$\mathcal{L}_\mu$ is a multi-modal logic with constructs for the extreme fixpoints of arbitrary definable monotonic mappings of type $\mathcal{P}(W) \to \mathcal{P}(W)$.

Syntax: propositional variables X, Y, ..., and constants p, q, ...

$$\varphi ::= p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle a \rangle \varphi \mid \mu X.\varphi.$$

X should have only positive occurrences in φ for $\mu X.\varphi$ to be well-formed.

Semantics: $M = \langle W, R, V \rangle$ like in PDL

$$[\![p]\!]_M = V(p),\ [\![X]\!]_M = V(X), \qquad [\![\neg\varphi]\!]_M = W \setminus [\![\varphi]\!]_M$$

$$[\![\langle \alpha \rangle \varphi]\!]_M = ([\![\alpha]\!]_M)^{-1}([\![\varphi]\!]_M), \qquad [\![\varphi \vee \psi]\!]_M = [\![\varphi]\!]_M \cup [\![\psi]\!]_M,$$

$$[\![\mu X.\varphi]\!]_M = \bigcap \{W' \subseteq W : [\![\varphi]\!]_{M^{W'}_X} \subseteq W'\}$$

$\mathcal{L}_\mu$ continued

Exercise 5 Prove that $\models_{\mathcal{L}_\mu} \mu X.\varphi \Leftrightarrow [\mu X.\varphi/X]\varphi$.

Exercise 6 Prove that if $\models_{\mathcal{L}_\mu} [\psi/X]\varphi \Rightarrow \psi$ implies $\models_{\mathcal{L}_\mu} \mu X.\varphi \Rightarrow \psi$.

Exercise 7 Prove that, for the same M, $[\![\langle a^*\rangle\varphi]\!]_M^{PDL} = [\![\mu X.\varphi \vee \langle a\rangle X]\!]_M^{\mathcal{L}_\mu}$.

Hint: Define $[\![\bigvee_{k<\omega} \varphi_k]\!]_M$ as $\bigcup_{k<\omega} [\![\varphi_k]\!]_M$ and prove that

$$[\![\langle a^*\rangle\varphi]\!]_M^{PDL} = [\![\mu X.\varphi \vee \langle a\rangle X]\!]_M^{\mathcal{L}_\mu} = [\![\bigvee_{k<\omega} \underbrace{\langle\alpha\rangle\ldots\langle\alpha\rangle}_{k \text{ times}}\varphi]\!]_M.$$

Greatest fixpoint is written as

$$\nu X.\varphi \rightleftharpoons \neg\mu X.[\neg X/X]\neg\varphi.$$

Exercise 8 Prove that the formula denoted by $\nu X.\varphi$ satisfies

$$[\![\nu X.\varphi]\!]_M = \bigcup\{W' \subseteq W : [\![\varphi]\!]_{M_X^{W'}} \supseteq W'\}.$$

Example: modelling board games in $\mathcal{L}_\mu$

$w \in W$ - the possible board configurations + whose turn is next;

$w_1 R w_2$ iff w_1 can change to w_2 by a single move;

$player_i$ - player i is to move next, $i = 1, 2$; win_i - player i wins.

$$M, w \models player_1 \vee player_2 \vee win_1 \vee win_2.$$

A single action $move$ with $R(move) = R$; $\Diamond \rightleftharpoons \langle move \rangle$, $\Box \rightleftharpoons [move]$.

Player 1 can win in one move from state w iff $M, w \models player_1 \wedge \Diamond win_1$.

Whatever player 2 does at state w, player 1 wins at the next move iff $M, w \models player_2 \wedge \Box \Diamond win_1$.

Player 1 has a winning strategy starting from state w iff

$$M, w \models \mu X.(win_1 \vee player_1 \wedge \Diamond X \vee player_2 \wedge \Box X).$$

Polyadic $\mathcal{L}_\mu$

$$[\![\mu X_1 \dots X_n.\varphi_1, \dots, \varphi_n]\!]_M =$$
$$\bigcap \{W_1' : W' \in (\mathcal{P}(W))^n, [\![\varphi_i]\!]_{M^{W_1', \dots, W_n'}_{X_1, \dots, X_n}} \subseteq W_i',\ i = 1, \dots, n\}.$$

Principle of Bekič $\mu X_1 X_2.\varphi_1, \varphi_2 \Leftrightarrow \mu X_1.[\mu X_2.\varphi_2/X_2]\varphi_2.$

Examlpe 1 Consider programs consisting of statements of the form

$$i:\ v_i := b_i;\ \textbf{if } c_i \textbf{ then goto } l_i' \textbf{ else goto } l_i''$$

where one of the labels $i = 1, \dots, n$, l_i', and l_i'' is $\mathbf{stop}$. Show that $\{P\}\mathbf{code}\{Q\}$ where $\mathbf{code}$ is of this form can be written as the formula

$$\begin{array}{lll} P \Rightarrow \mu X_1 \dots X_n X_{\mathbf{stop}} & . & [a_{v_1:=b_1}]((c_1 \Rightarrow X_{l_1'}) \wedge (\neg c_1 \Rightarrow X_{l_1''})), \\ & & \dots \\ & & [a_{v_n:=b_n}]((c_n \Rightarrow X_{l_n'}) \wedge (\neg c_n \Rightarrow X_{l_n''})), \\ & & Q. \end{array}$$

Embedding PDL into $\mathcal{L}_\mu$

$$\mathsf{t}(\bot) \rightleftharpoons \bot \qquad \mathsf{t}(p) \rightleftharpoons p$$

$$\mathsf{t}(X) \rightleftharpoons X \qquad \mathsf{t}(\varphi \Rightarrow \psi) \rightleftharpoons \mathsf{t}(\varphi) \Rightarrow \mathsf{t}(\psi)$$

$$\mathsf{t}(\langle Id \rangle \varphi) \rightleftharpoons \mathsf{t}(\varphi) \qquad \mathsf{t}([Id]\varphi) \rightleftharpoons \mathsf{t}(\varphi)$$

$$\mathsf{t}(\langle a \rangle \varphi) \rightleftharpoons \langle a \rangle \mathsf{t}(\varphi) \qquad \mathsf{t}([a]\varphi) \rightleftharpoons [a]\mathsf{t}(\varphi)$$

$$\mathsf{t}(\langle \varphi? \rangle \psi) \rightleftharpoons \mathsf{t}(\varphi) \wedge \mathsf{t}(\psi) \qquad \mathsf{t}([\varphi?]\psi) \rightleftharpoons \mathsf{t}(\varphi) \Rightarrow \mathsf{t}(\psi)$$

$$\mathsf{t}(\langle \alpha \cup \beta \rangle \varphi) \rightleftharpoons \mathsf{t}(\langle \alpha \rangle \varphi) \vee \mathsf{t}(\langle \beta \rangle \varphi) \qquad \mathsf{t}([\alpha \cup \beta]\varphi) \rightleftharpoons \mathsf{t}([\alpha]\varphi) \wedge \mathsf{t}([\beta]\varphi)$$

$$\mathsf{t}(\langle \alpha ; \beta \rangle \varphi) \rightleftharpoons \mathsf{t}(\langle \alpha \rangle \langle \beta \rangle \varphi) \qquad \mathsf{t}([\alpha ; \beta]\varphi) \rightleftharpoons \mathsf{t}([\alpha][\beta]\varphi)$$

$$\mathsf{t}(\langle \alpha^* \rangle \varphi) \rightleftharpoons \mu X.\mathsf{t}(\varphi) \vee \mathsf{t}(\langle \alpha \rangle X) \qquad \mathsf{t}([\alpha^*]\varphi) \rightleftharpoons \nu X.\mathsf{t}(\varphi) \wedge \mathsf{t}([\alpha]X)$$

The clause $\mathsf{t}(X) \rightleftharpoons X$ in the definition is needed for the handling of formulas such as $\langle \alpha \rangle X$ which commence in the translation of Kleene star.

Embedding CTL into $\mathcal{L}_\mu$

A single primitive program – for the passage of a unit of time.

$$\mathsf{t}(\bot) \rightleftharpoons \bot, \ \mathsf{t}(p) \rightleftharpoons p$$

$$\mathsf{t}(\varphi \Rightarrow \psi) \rightleftharpoons \mathsf{t}(\varphi) \Rightarrow \mathsf{t}(\psi)$$

$$\mathsf{t}(\exists \circ \varphi) \rightleftharpoons \langle - \rangle \mathsf{t}(\varphi)$$

$$\mathsf{t}(\exists(\varphi \mathsf{U} \psi)) \rightleftharpoons \mu X.\psi \vee (\mathsf{t}(\varphi) \wedge \langle - \rangle X)$$

$$\mathsf{t}(\forall(\varphi \mathsf{U} \psi)) \rightleftharpoons \mu X.\psi \vee (\mathsf{t}(\varphi) \wedge [-] X)$$

The End