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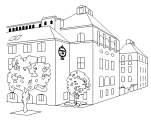
November 11, 2005

# 2D1253, Numerical Algebra



## Reading instructions and Review Questions:

The following instructions and questions are intended to be a help when reading the course and preparing for exam. Among the questions, some may be answered just by reading the text, while some need some hand computation. Even if MATLAB is helpful when preparing the course, only hand calculation is needed for these questions.

Instructions refer to sections in the Demmel text book (D), the Eigentemplate book (E) or my lecture notes, Topics in Numerical linear Algebra, (R). The questions are numbered in one sequence.

D 1.7 Vector and Matrix norms. Part of this can be found in mathematics text books.

**Q 1.** State the 3 conditions that a vector norm $\|x\|$ must satisfy!

**Q 2.** Prove that $\|x\|_p \equiv (\Sigma_i |x_i|^p)^{\frac{1}{p}}$ defines a vector norm when $p \geq 1$ but not when $p < 1$.

**Q 3.** State the 4 conditions that a consistent matrix norm $\|A\|$ must satisfy!

**Q 4.** Prove that the operator norm $\|A\| \equiv \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$, for any vector norm $\|x\|$, defines a matrix norm. This is called the matrix norm induced by, or subordinate to, the vector norm $\|x\|$.

**Q 5.** Show that the operator norm induced by the Euclidean vector norm $\|x\|_2$ is $\|A\|_2 = \sigma_1(A)$, the largest singular value.

**Q 6.** What are the singular values of an orthogonal matrix?

**Q 7.** Prove that if $\|X\| < 1$ in any consistent (submultiplicative) matrix norm, the matrix $I - X$ is nonsingular, and $\|(I - X)^{-1}\| \leq \frac{1}{1 - \|X\|}$!

**Q 8.** What is the condition number $\kappa(A)$ of a matrix $A$ (with respect to inversion)?

**Q 9.** Show that $\|I + hA\| \leq 1 + |h|\|A\|$ for any operator norm!

**Q 10.** Show that the condition number $\kappa(I + hA)$ approaches 1 for $h \to 0$!

D 2.2 Linear Systems, perturbation theory.

**Q 11.** Derive an expression for the perturbation of the solution $x$ of a linear system $Ax = b$ when the right hand side $b$ is perturbed.

**Q 12.** For which perturbation $\delta b$ does $\delta x$ get maximal norm, giving equality in the perturbation bound? Use the Euclidean vector norm and singular value decomposition!

**Q 13.** Derive an expression for the perturbation of the solution $x$ of a linear system $Ax = b$ when the matrix $A$ is perturbed.

**Q 14.** How large is the smallest perturbation that makes the matrix $A$ singular? Use the singular value decomposition to find one such perturbation!

**Q 15.** What is the componentwise relative condition number $\kappa_{CR}(A)$.

**Q 16.** Mention a class of matrices for which $\kappa_{CR}(A)$ is significantly smaller than the standard condition number.

D 2.4 Rounding errors in Gaussian elimination.

**Q 17.** Give a bound on the elements of the perturbation $E$ caused by rounding errors during Gaussian elimination $A + E = LU$, where $L$ and $U$ are the factors computed with floating point arithmetic.

**Q 18.** What is the maximal growth factor during Gaussian elimination with partial row pivoting?

D 5.2 Perturbation theory for eigenvalues.

**Q 19.** Formulate the Courant Fischer minimax theorem for real symmetric matrices.

**Q 20.** Show that a perturbation $E$ to a real symmetric matrix moves the eigenvalues at most $\|E\|_2$ away.

**Q 21.** Show that, if $B = A + M$ where $M$ is positive definite, then $\lambda_i(B) \geq \lambda_i(A)$.

**Q 22.** Show that $\lambda_{\min}(Q^T AQ) \geq \lambda_{\min}(A)$ where $Q$ is an $n * k$ matrix with orthonormal columns.

D 6.6.1 R4.2 Krylov subspaces, Arnoldi algorithm.

**Q 23.** What is a Krylov subspace?

**Q 24.** The Arnoldi algorithm has a basic recursion

$$AQ_k = Q_k H_{k,k} + R$$

discuss properties of the basis $Q_k$, the reduced matrix $H_{k,k}$ and the residual $R$!

**Q 25.** Show that if $\theta$ is an eigenvalue and $s$ is an eigenvector of $H_{k,k}$, then $y = Q_k s$ gives an approximate eigenvector of $A$. What is the Euclidean norm and direction of its residual $r = Ay - y\theta$?

**Q 26.** Describe the Krylov algorithm.

**Q 27.** Show how the Arnoldi algorithm can be derived from the Krylov algorithm by means of a QR factorization.

**Q 28.** Show that, when Arnoldi is applied to a real symmetric matrix, one gets the Lanczos algorithm.

E 4 E 7 Lanczos for eigenvalues.

**Q 29.** What happens when Lanczos is applied to a matrix with multiple eigenvalues?

**Q 30.** Describe full and selective reorthogonalization. In which cases should one choose one or the other of these?

**Q 31.** Describe shift invert spectral transformation. In which cases is it applied?

**Q 32.** Describe implicit restart for the Arnoldi algorithm.

**Q 33.** What is break down in the nonsymmetric Lanczos algorithm?

**Q 34.** When is nonsymmetric Lanczos to be preferred to Arnoldi?

R 5 D 6.6.2-6 Linear systems, iterative algorithms.

**Q 35.** Show how the Arnoldi algorithm can be used to get approximate solutions to a linear system of equations.

**Q 36.** Show how the GMRES (Generalized Minimal Residual) algorithm is one variant of the answer of previous question!

**Q 37.** Show how the Lanczos algorithm is used to solve a system with a symmetric matrix $A$!

**Q 38.** Show that the conjugate algorithm can be derived from the Lanczos algorithm when the matrix $A$ is positive definite!

**Q 39.** What is meant with that two vectors are $A$ conjugate?

**Q 40.** Show that the search directions $p_k$ in the conjugate gradient algorithm are $A$ conjugate!

**Q 41.** Show that the residuals $r_k$ in the conjugate gradient algorithm are orthogonal to each other!

**Q 42.** What is a preconditioning of a matrix?

**Q 43.** Describe incomplete Cholesky preconditioned conjugate gradient, ICCG. What is meant by ICCG(0) and ICCG(1)?

**Q 44.** Show that the nonsymmetric (two sided) Lanczos algorithm applied to the matrix $A$ of a linear system $Ax = b$ leads to the Quasi Minimal Residual (QMR) algorithm!

**Q 45.** Give some advantages and disadvantages of QMR compared to GMRES!

E 6 Computing the Singular Value Decomposition.

**Q 46.** Given a $m \times n$ matrix $A$. Define the $(m+n) \times (m+n)$ Hermitian matrix

$$H(A) \equiv \begin{bmatrix} 0 & A \\ A^H & 0 \end{bmatrix}$$

Show that the eigenvalues of $H$, $\lambda(H) = \pm\sigma(A)$, plus and minus the singular values of $A$ and $|m-n|$ zero eigenvalues! Which are the eigenvectors of $H$?

**Q 47.** Show that the Lanczos algorithm applied to $H(A)$, starting at the vector $x_1 = (0, v_1^T)^T$ of order $m+n$, gives one Golub Kahan bidiagonalization of $A$!

**Q 48.** Show that starting at $x_1 = (u_1^T, 0)^T$, we get another bidiagonalization.

**Q 49.** Describe the LSQR iteration for an overdetermined linear system

$$min_x \|Ax - b\|_2$$

*Good Luck!*