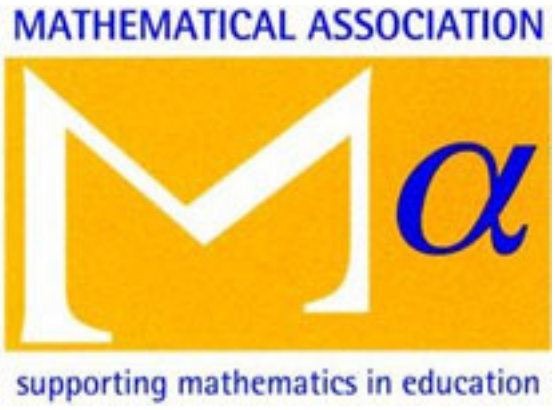

1936: Post, Turing and 'a kind of miracle' in mathematical logic

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Preamble

In the 1930s several mathematicians, principally Alonzo Church (1903-1995), Stephen Kleene (1909-1994), Emil Post (1897-1954) and Alan Turing (1912-1954), began investigating the notion of *effective calculability*. (A function from natural numbers to natural numbers is *effectively calculable* if there is some finite rule or mechanism which will calculate the value of the function for any natural number.) Central to this activity was the notion of recursiveness. Loosely, *recursion* is a process of defining a function by specifying each of its values in terms of previously defined values. (The sequence of Fibonacci numbers, $F(1)$, $F(2)$, $F(3)$, ..., for example, can be defined by setting $F(1) = 1$, $F(2) = 1$, and then for all natural numbers $n > 2$, $F(n) = F(n - 1) + F(n - 2)$, so that $F(3) = 2$, $F(4) = 3$, $F(5) = 5$, etc.). Within 10 years a new branch of mathematical logic, *Recursive Function Theory* (RFT), had been established (for a brief overview of RFT see [1]).

Calculation by means of recursion has a long history. Richard Dedekind, however, appears to have been the first to use the concept of (primitive) recursion (he called it *definition by induction*) explicitly in his exposition of the theory of natural numbers (*Was sind und was sollen die Zahlen?* (1888)). But it was Thoralf Skolem in his 1923 paper [2] who, in demonstrating that many number-theoretic functions are primitive recursive, grasped clearly and decisively the full power of the recursive mode of thought in his formulation of a portion of elementary arithmetic. Again, Kurt Gödel, in his epoch-making paper of 1931, *On formally undecidable propositions of Principia Mathematica and related systems*, gave a precise definition of a primitive recursive function. Then, in a lecture given in 1934, Gödel, modifying a suggestion of Jacques Herbrand, proposed a definition of *general recursiveness*; he was proposing an answer to the question of 'what one would mean by "every recursive function".' Meanwhile, during 1932-35, Church and his student Kleene, had given an exact definition of a class of computable number-theoretic functions which they called *λ-definable* which seemed to embrace the notion of *effective calculability*. Indeed, almost simultaneously with Gödel's 1934 lecture Church was to propose his famous thesis, that every effectively calculable function is *λ-definable*. In 1935 Church chose to express his thesis in terms of *Herbrand-Gödel general recursiveness*. It did not take the Princeton group, Church, Kleene and J. Barkley Rosser, long to realise that λ-definable and Herbrand-Gödel recursive are equivalent; between them Church and Kleene proved the equivalence. Gödel, however, was unwilling to endorse the equivalence of effective calculability either with λ-definability or

recursiveness. But now we are at the threshold of that remarkable year, 1936, which Gödel, in his 1946 [3], speaks of a 'kind of miracle' that the class of processes which can be accomplished by mechanical means can be given a precise mathematical characterisation. So what finally convinced Gödel?

Turing and Post died 50 years ago and this note commemorates their contributions not only to our understanding of algorithms but also to the enormous influence, out of all proportion to their published output, they have had on the 'high-tech' society in which we live. By 1936 Post was a seasoned logician although, as we shall see, he held back from publishing much of his research in the 1920s. Turing's involvement in our story was sudden and motivated initially, not by RFT but by the Hilbert challenge of the *Entscheidungsproblem* (*decision problem*), the problem of determining whether or not a given formula of the predicate calculus (PC) is valid.

Emil L. Post

Emil Post, born of Jewish parents in Augustów, Poland, arrived in New York in 1904 where he lived for the rest of his life. There he attended a school for talented students which was on the same campus as City College which was to feature so much in his life, a life dogged by misfortune and illness. At the age of 12 he lost his left arm in a tragic accident which thwarted his ambition to be an astronomer. Throughout his adult life Post suffered from periods of crippling manic-depressive illness which tended to erupt when he was at his most creative. There was also, especially in the 1920s, an almost total neglect of the field of logic in America so, a fortiori, there would have been precious little understanding of what Post was attempting at that time. This partly explains why so much of these early researches remained unpublished for some 20 years. Post also had to face the daunting prospect of having to earn his living teaching in public high schools which he did for the most part until 1935 when he was rescued by his alma mater, City College, where he was on the mathematical faculty until his death. Even so the teaching load of 16 contact hours per week was formidable; individual faculty offices and secretarial help were unavailable so Post retreated to his home to do his research. His young daughter Phyllis was expected to remain silent. In later life, Phyllis Goodman, as she was to become, described her father as a genius and her mother as a saint [4, p. xii]. She explained how her mother typed her father's manuscripts and correspondence, handled financial matters and generally buttressed him from any intrusion into his daily regime.

Post took his B. S. degree at City College in 1917 having majored in mathematics. Although we think of him primarily as a logician his earliest interest was in analysis. As an undergraduate he worked out a generalisation of the differential operator D^n when n is not an integer and the resultant paper featured an important result about inverting the Laplace transform. As a graduate student at Colombia University, where Post had enlisted in 1917, he also published a brief paper on the functional equation

of the Gamma function but his longest paper [4] established the equivalence of the theory of polyadic groups (groups with $n \geqslant 2$ arguments where $n = 2$ is the standard case) to the theory of finite cyclic extensions of ordinary groups. However, it was in his doctoral thesis, which was to prove so influential, that he proposed a framework for systems of logic as combinatorial mathematical calculi; the most powerful mathematical or logical system is essentially a set of rules determining how certain strings of symbols can be transformed into other strings of symbols. Such systems Post referred to as being obtained by 'generalisation by postulation' but later as being in *canonical form A*. The monumental *Principia Mathematica* (PM) of Russell and Whitehead was the backdrop to Post's programme. In his thesis Post extracted the *sentential calculus* (SC) part of PM, treated it as a combinatorial calculus, showed that its axioms were complete and consistent and, by means of the truth-table algorithm, his innovation, also solved the decision problem for SC. Post also extended his truth-table method for two truth values to an arbitrary finite number of truth values. Post's doctoral thesis marks the beginning of what soon became a major area of Mathematical Logic, namely *Proof Theory*.

The main thrust of Post's thinking during the year 1920-21, as post-doctoral fellow at Princeton University, was to show that a very wide class of formal logical systems can all be expressed in a particular form which he called *canonical form*. He considered three such forms A, B, C, subsequently proved by him to be equivalent and, for example, showed that the PC part of PM could be put into form B and hence into canonical form C. Post believed that his techniques could be used to show that all provable formulas of PM could be regarded as the set of strings generated by some system in canonical form C. The basic idea was that of a *canonical production* which takes the form

$$g_0P_1g_1P_2\ldots\ g_{m-1}P_mg_m \rightarrow h_0P_{i_1}h_1P_{i_2}h_2\ldots\ P_{i_n}h_n.$$

where the g_i are fixed strings, which may be null, defined on a finite alphabet Σ, the P_i are variable strings and the subscripts $i_1\ldots\ , i_n$ are to be found among 1, 2, ... , m, and need not be distinct. *Modus ponens*, for example, which can be expressed by the schema:

$$\frac{\mathrm{A} \qquad \mathrm{A} \Rightarrow \mathrm{B}}{\mathrm{B}}$$

could be written in Post's canonical form as

$$g_1Pg_1$$
$$g_1Pg_2Qg_1$$
produce
$$g_1Qg_1$$

where g_1 represents the empty string, g_2 represents the string consisting of the single symbol $\Rightarrow$ and P, Q represent any strings regarded as well-formed

formulas. Readers may check that for $\Sigma = (a, b, c)$ and axioms a, b, c, aa, bb, cc all palindromes are generated by the productions $P \rightarrow aPa$; $P \rightarrow bPb$; $P \rightarrow cPc$.

Still at Princeton, Post proved a beautiful theorem, the *normal-form theorem*, which showed how a system in canonical form C could be reduced to the astonishingly simple form involving one initial assertion (axiom) and each production taking the (normal) form $gP \rightarrow Ph$. An application of Cantor's diagonal method led Post to the conclusion that the decision problem for normal systems and hence of PM has a negative solution but, as he was to remark in [5], 'the correctness of this result is clearly entirely dependent on the trustworthiness of the analysis leading to the above generalisation', namely to *Post's thesis*, which can be stated as: any finitely given language is generated by rules of some canonical normal system. Post concluded in [5] that a complete symbolic logic is impossible.

Again, in 1921, Post was led to a class of apparently simple but frustrating problems called *tag*. A *tag system* is a normal system in which the antecedent strings g have the same length $|g|$ and the consequent string h depends only on the first symbol of the associated g. The 'simple' case: $agP \rightarrow Paa$; $bgP \rightarrow Pbbab$ (g is any string from $\{a, b\}$ and $|g| = 2$) 'proved intractable'. One variant of the problem is to devise an algorithm to determine whether the process terminates with the empty string. Post conjectured that tag would be a candidate for recursive unsolvability. He was right; tag was later proved unsolvable by Marvin Minsky in 1961.

From the above considerations it is clear that Post had anticipated the salient results of the 1930s principally by Gödel, Church and Turing. He apparently lectured on the incompleteness of PM at Colombia University in the 1920s but he was to concede that his work was 'fragmentary'. Significantly, Post did not publish his results. He was unhappy at Princeton and at the end of 1921 he collapsed with the first attack of manic depression which necessitated hospitalisation. His [5] was submitted to the *American Journal of Mathematics* in 1941 and was rejected by the editor Hermann Weyl who, in mitigation, wrote in a letter to Post that 'you may be comforted by the certainty that most of the leading logicians, at least in this country, know in a general way of your anticipation'. Subsequently a much shorter paper [6], containing only the normal form theorem, was accepted. His original submission which eventually appeared in the 1965 anthology [5] contained a philosophical appendix based on a diary Post had kept since 1916. We can but imagine the anguish Post suffered on the appearance of Gödel's 1931 incompleteness paper and the announcement by Church in 1935 of an unsolvable problem in elementary number theory.

Post's 1936 paper [7] which appeared in the first issue of the Journal of Symbolic Logic showed that his creativity had not been blunted by his disappointments. Independently of Turing but aware of Church's Thesis, Post gives an analysis of the computing process very similar to that given by Turing but there is a subtle difference. Whereas Turing concentrated on the mechanics of the machine, its internal configuration; Post focused on the

instructions or 'software' that would make the 'machine' work. We note that Post in [5] and [6] gave a characterisation of effectiveness in terms of his canonical and normal systems.

In possibly his most influential paper [8], Post formulated a method of developing the concept of a *recursively enumerable* (r.e.) set independently of the theory of computable functions [see Appendix]. By extending the r.e. concept to include the empty set Post showed for the first time that recursive sets are r.e. sets with r.e. complements. He also showed that every infinite r.e. set contains an infinite recursive subset and that there exists an r.e. set that is not recursive. He also set out an elegant 'miniature' form of Gödel's incompleteness theorem. However, what can be regarded as the most original contribution to this paper is his treatment of the ways in which one r.e. set could be considered reducible to another. An r.e. set X is said to be many-one reducible to an r.e. set Y if and only if there exists a recursive function f such that $x \in X \Leftrightarrow f(x) \in Y$. If the function is one-one then we have one-one reducibility. Post proved the existence of an r.e. set K which is *complete* with respect to many-one (one-one) reducibility in the sense that every r.e. set X is many-one (one-one) reducible to K. Hence K has *highest degree of unsolvability* with respect to many-one (one-one) reducibility. (We meet a complete set in the context of the *halting problem* in the appendix.) Post formulated several mathematical reducibility concepts, the most general of which he takes from Turing 1939 [9].

Post now asked whether there exist r.e. sets of different degrees. If we denote by $\mathbf{0}$ the unique lowest degree of the recursive sets and let $\mathbf{0}'$ denote the degree of K, then $\mathbf{0} < \mathbf{0}'$. *Post's Problem*, as it became known, is to locate an r.e. degree $\boldsymbol{a}$ such that $\mathbf{0} < \boldsymbol{a} < \mathbf{0}'$. His *simple* sets whose complements $\bar{S}$ do not contain infinite r.e. sets did not provide the answer. Progress was made, however, by Post himself in the abstract [10] in which degrees of unsolvability less than $\mathbf{0}'$ were obtained but not for r.e. sets. Friedberg and Mucnik independently and almost simultaneously solved the problem in 1956 [11, 12].

In [13] and [14] Post established the recursive unsolvability of two problems in combinatorial mathematics, the *Post Correspondence Problem* and the word problem for the semi-groups posed by Axel Thue in 1914. A Post *correspondence system* consists of a finite alphabet A and a finite set of ordered pairs of strings (g_i, h_i), $1 \leqslant i \leqslant m$, where a word u on A is called a solution of the system if for some sequence $1 \leqslant i_1, i_2, \ldots, i_n \leqslant m$ (the i_j need not be distinct) we have,

$$u = g_{i_1}g_{i_2}\ldots g_{i_n} = h_{i_1}h_{i_2}\ldots h_{i_n}.$$

The Post correspondence problem is to devise an algorithm for determining whether a given Correspondence system has a solution. The paper [13], in which Post shows that no such algorithm exists, begins with the unsolvability of the decision problem for Post normal systems.

In the word problem for semi-groups consider strings, possibly null, defined on a finite alphabet and suppose (u_1, v_1), (u_2, v_2), ..., (u_n, v_n)

represents a finite list of pairs of strings, or words, each pair of words u_i, v_i deemed to be *equivalent* (think of the list as a dictionary). We call two arbitrary words (u, v) equivalent if v can be obtained from u by a finite number of transformations each of which consists of a replacement in a given word w of a segment identical with u_i by the word v_i or vice-versa, $1 \leqslant i \leqslant n$. The word problem for semi-groups is to find an algorithm to show for an arbitrary pair (u, v) whether or not u is equivalent to v. Post's strategy was to *reduce* the problem to that of the halting problem for Turing machines known to be unsolvable (see appendix). Thus the solution to this word problem is negative. Post had qualms about relying solely on Turing's work, however, and as well as fashioning an independent proof, he presented a thorough critique of Turing's 1936-37 paper [17] which contained a number of errors. Thus Post was the first to prove the unsolvability of a 'classical' mathematical problem which was not specifically related to logic. We note, in passing, that Andrej A. Markov, independently, arrived at the same result by exploiting Post normal systems!

We learn from the distinguished logician Martin Davis, one of Post's students, that Post's classes 'were tautly organized tense affairs' [4, p. xxv]. Even so Post was popular, inspiring and very effective with both strong and weak students. He was a 'stickler for care and precision in mathematical discourse'.

When Post was 50 his health improved but in 1954 he broke down again and was institutionalised. Still no drug therapy was available; electro-convulsive shock treatment was administered and a short while after receiving it, still in hospital, he died suddenly of a coronary.

Alan Mathison Turing

Alan Turing was born in a nursing home in Paddington, London, on 23 June 1912 and was raised mainly by relatives until he was of school age. After a prep. school start to his education he was accepted at Sherborne, 'a moderately distinguished public school' [15], in 1926. From boyhood Turing, as well as being especially keen about chemistry, was interested also in mathematics, theoretical physics and astronomy. Something of a maverick, he was keen to conduct his own experiments and to work things out from first principles. He was imaginative, sceptical and inquisitive with a sense of humour. From having a sunny disposition as a young boy he became somewhat withdrawn and awkward as an adolescent. His interactions with authority were often, given his character, difficult and bumpy affairs. In the sixth form at Sherborne he was supervised in mathematics by a young teacher, Donald Eperson, well known to Mathematical Association members, who was recently down from Oxford. Eperson had the insight and shrewdness to leave such a mathematical talent to his own devices and to give assistance only when necessary. In spite of his unorthodox methods, lack of polish, proneness to trivial errors and the near illegibility of his handwriting Turing won a scholarship to King's

College, Cambridge in 1930. In 1934 he graduated with distinction in the mathematical Tripos examination. Less than a year later he was elected to a Fellowship of the college, his dissertation having been a proof of the Central Limit Theorem which amounted to a rediscovery.

In 1935 Turing attended a course on the Foundations of Mathematics given by the topologist Max H. A. Newman. Although he was interested in mathematical logic he had been working in other areas of mathematics, particularly group theory. The lectures considered such concepts as the consistency, completeness and decidability of various formal axiomatic systems and the Gödel incompleteness results but it was the *Entscheidungsproblem* which captured his attention and dominated his thinking from the Summer of 1935 until the early Spring of 1936. David Hilbert called this problem the *fundamental theorem of mathematical logic* for he surmised that an algorithmic solution to it would entail that any mathematical problem would be decided by an algorithm. Thus, if there is a mathematical problem that is algorithmically unsolvable then the unsolvability of the Entscheidungsproblem would follow. Soon Turing saw that the problem had a negative solution but what was required now was a precise mathematical analysis of the informal concept of calculability by a strictly mechanical process; it was necessary to survey the class of all possible algorithms. This is what Turing achieved and the tools he fashioned for the purpose have become fundamental for the development of computer science. In mid-April 1936 he submitted a draft of his paper *On Computable Numbers, with an Application to the Entscheidungsproblem* to Newman who, at first, was sceptical of Turing's analysis; he was astonished that so simple a concept as that of a *Turing machine* (Church's label) could deliver the answer to such an outstanding problem. Eventually, though, Newman was persuaded and encouraged Turing to publish his paper. It is a remarkable *tour de force* especially as he was so new to this field. Typically, Turing's success seems to have depended on a mind uncluttered by the work of others and on his view of symbolic logic as a branch of applied mathematics which induced a physical, even engineering, edge to his thinking. There were technical errors in his paper, as Post discovered, but these could be rectified.

We shall not describe a Turing machine here (but see [1, p. 291]). Suffice it to say that Turing, beginning with a human agent carrying out a pen and paper calculation, proceeds by a process involving simplifications and the elimination of irrelevant details to arrive at the familiar finite state device on a one-way infinite tape. Turing's paper includes the unsolvability of the halting problem and of the Entscheidungsproblem as well as a construction of a universal Turing machine capable of mimicking the behaviour of any Turing machine. We see here the genesis of the concept of the stored-program computer which later von Neumann was to exploit (see Appendix). Turing also proved that there is no algorithm to determine whether a Turing machine starting with a blank tape will ever print some particular symbol. It was this problem, closely allied to the halting problem,

which Turing expressed in the language of PC and thus obtained the unsolvability of the Entscheidungsproblem.

However, just as Turing was presenting the draft of his paper to Newman, Church submitted a paper [16] with a simple proof of the unsolvability of the decision problem for PC. The daunting news of Church's achievement reached Cambridge in May 1936 and seemed to preempt Turing's analysis of calculability and his Entscheidungsproblem result. However, as Turing's characterisation of calculability was sufficiently different from that of Church, his paper was submitted after all and appeared later with an added appendix which sketched a proof of the equivalence of *his* notion of calculability with that of λ-definability. Indeed, this paper represents something of a coup as Gödel was finally convinced that the Turing machine model of computation therein was of fundamental importance in establishing the validity of Church's thesis. Turing had, in a sense, outflanked those at Princeton who had been working for some years on the problem of effectiveness.

Turing, supported by his fellowship funds from King's, decided to spend the year 1936-37 in Princeton and arrived there at the end of September 1936. Eventually this was extended to 1938 so that he could do Ph.D. work under the supervision of Church who had, incidentally, refereed Turing's paper [17] which appeared in print in January 1937. For the dissertation Turing was to investigate Church's idea of ordinal logics in the context of Gödel's incompleteness theorems. The upshot was a profound, difficult and important paper [9] in which Turing investigates the possibility of escaping Gödel's incompleteness theorem by substituting a single given logic by a system of logics (ordinal logics) derived from each other by transfinite iterations. More importantly, almost incidentally, Turing introduces the concept of an *oracle* which enabled a classification of unsolvable problems and led to a rich theory of *relativised calculability* and, in turn, to modern *relativised complexity theory*. An Oracle machine (o-machine) is a Turing machine having access to an oracle which can, as if by magic, perform a non-computable operation in one step; such a machine is not purely mechanical. Newman likened an oracle to a mathematician having an idea as opposed to a mathematical method. To fix ideas we now define a set A as recursive in B, denoted by $A \leqslant B$, if and only if there is an o-machine which decides membership of A using B as an oracle. We say a set A is reducible to a set B if $A \leqslant B$. The collection of sets X such that $A \leqslant X$ and $X \leqslant A$ is called the *degree* (Turing degree or degree of unsolvability) of A. Clearly a degree is an equivalence class and degrees are partially ordered in the sense that $d_1 > d_2$ if and only if sets in the equivalence class d_1 are more non-recursive than sets in the equivalence class d_2. Turing did nothing further with his idea of o-machines but Post in [8], as we indicated earlier, took it as his basic notion for a theory of degrees of unsolvability and duly credited Turing with the result that for any problem about integers there is another of higher degree of unsolvability. Yet, again, Turing had hit the target; his idea opened up the entire subject of generalised recursion theory with present day

ramifications. His thesis, submitted in May 1938, was judged 'excellent' by the committee who conducted the viva and in June 1938 Turing received his PhD.

Turing returned to Cambridge in July 1938 even though von Neumann had offered him a position as his assistant at the Institute for Advanced Study with a healthy stipend. In 1939 he participated in Wittgenstein's classes devoted to the foundations of mathematics but, alas, no record of the discussions survives. By the time the Second World War began Turing had already been recruited to an ongoing project to break the codes used in German military communications. Already, at Princeton, possibly in connection with his growing concern with cryptanalysis, he designed and constructed various parts of an electro-mechanical binary multiplier. Turing also had ideas for the design of an 'analogue' device for investigating the distribution of the zeros of the Riemann zeta-function, a project never completed due to the intervention of the Second World War.

On the fourth of September, the day after Britain entered the war, Turing reported to Bletchley Park where he was to become the chief scientific figure in the British cryptological effort with particular responsibility for deciphering communications between German submarines and their home bases. It was work to which Turing was eminently suited given his theoretical and practical talents. We shall not pursue the Bletchley story here; it has been comprehensively recorded in, for example, [15, 18, 19]. Suffice it to say that a crucial role was to be played by Turing whose invention of an electro-mechanical machine called a Bombe was to revolutionise the way in which the German Enigma code could be broken. Gordon Welchman, another brilliant Cambridge mathematician, realised how Turing's prototype could be adapted to enhance greatly its performance. Yet another improvement by Turing combined with Welchman's suggestion were incorporated into an updated version of the original Bombe which was called 'the spider'. It worked! However, even these modified Bombes would be inadequate if the Germans introduced greater complexity into their procedures. This led in 1943 to the construction of the world's first electronic automatic calculator device called Colossus which used vacuum tube circuits to carry out complex Boolean calculations rapidly. Thus when the war ended Turing had a sound basic knowledge of electronics and understood that computing machines could be constructed using electronic circuits.

Throughout the war Turing continued to think of constructing a universal computer, a machine that, if realized, could play chess, solve jigsaw puzzles and even exhibit intelligent behaviour. He was interested in something more than a device capable of very rapid calculation. In short, he envisaged building a 'brain'. Soon, the National Physical Laboratory (NPL) enlisted Turing to design its computer, known as the Automatic Computing Engine (ACE). By the end of 1945 he produced his remarkable ACE report which presented a design for a machine that called for comparatively little hardware but which, therefore, put a greater burden on those writing

programs. Unfortunately liaison between Turing and the engineers left much to be desired. After the initial designing his main task was to write programs for ACE but these inevitably relied on the engineering details so that when engineering difficulties arose, such as the problem of storage, changes of plans ensued. It needs to be remembered, too, that the work at NPL was overshadowed by more powerfully supported American projects. Increasingly frustrated Turing quit NPL in 1948 to join, at Newman's invitation, the computer laboratory at Manchester University.

In 1946, as we know now, Turing was ahead of the field. His understanding of such issues as microprogramming and the use of a stack for a hierarchy of subroutine calls bore testimony to this. It was, however, von Neumann who was to be given the credit for the basic architecture of the modern computer. In his 1945 preliminary report on EDVAC (Electronic Discrete Variable Computer) von Neumann clearly envisaged an 'all-purpose' computer although the emphasis was on numerical calculation. Turing's influence on the report is clearly discernible so it is rather curious that his name does not appear in it especially as von Neumann must have been aware of Turing's concept of a universal machine by 1939 at the latest. It was much later that Turing received due recognition for his contributions. One can imagine how bitter he felt about the ineptitude of the NPL management that had been so prodigal with his talent and had thwarted his aspirations as revealed in his ACE report.

At Manchester, virtually sidelined from the development in automatic computation, Turing did produce a short paper in 1949 entitled '*Checking a large routine*' which anticipates ideas of program proof not developed until the 1960s. With time on his hands he became an accomplished marathon runner and launched himself into a variety of projects. He proved the unsolvability of the word problem for semi-groups with cancellation. Then his thinking took a philosophical turn with the controversial *Computing machinery and intelligence* (1950), published in the leading philosophical journal *Mind*, which addressed the question 'Can machines think?' and featured the famous *Turing Test*. Roughly, if responses from a computer when interrogated (via a mechanical link) were indistinguishable from that of a human then the computer could be said to be thinking; deemed to have passed the test. The Artificial Intelligence debate continues among mathematicians and philosophers to this day. In 1951 Turing was elected FRS for his 1936 work.

As a schoolboy, Turing had read and been inspired by D'Arcy Thompson's book *On Growth and Form* [20]. In 1952 he published in the Philosophical Transactions of the Royal Society a paper entitled *The Chemical Basis of Morphogenesis* [21], which turned out to be a founding paper of modern non-linear dynamical theory. Turing's term 'Morphogenesis' is now more understandably referred to as 'pattern formation'. Turing himself did further important work in the paper *A Diffusion Reaction Theory of Morphogenesis in Plants* [22]. This remained

unpublished for many years after his death, and only finally appeared in the third volume of Turing's *Collected Works* [23].

The last two years of Turing's life were blighted by his public exposure as a homosexual following an act of betrayal. These were pre-Wolfenden times. To prevent his incarceration the judge directed that Turing should receive oestrogen injections to curb his sex drive. Turing was deemed a security risk; as a result he was disqualified from continuing secret cryptological work. There is evidence, too, that Turing, who had done so much for his country, was hounded by the governing authorities of the land. Whatever, on June 7, 1954, he died having bitten into an apple impregnated by cyanide. The official verdict was suicide.

The Post-Turing Legacy.

In the 1980s information about Turing's vital role in decrypting German communications during the war entered the public domain. A play, *Breaking the Code*, by Hugh Whitmore, first staged in London in 1986, and later adapted for television, faithfully portrayed his Bletchley contributions, the importance of his mathematical ideas and the problems which beset him towards the end of his life. In 1999 *Time* magazine judged him among the twenty greatest scientists and thinkers of the 20th century. Bletchley is now open to the public. As yet there is no artefact capable of passing the Turing test but Turing would have been pleased that Gary Kasparov, the world chess champion and arguably the strongest player of all time, was beaten by a machine, Deep Blue, in 1997.

And what of Post? The development of mathematical logic was profoundly influenced by him, but it was only in 1994, when his collected works appeared [4], that Post duly received the credit for his contributions. His theory of degrees led to a spate of papers on the subject and in his study of various kinds of recursive reducibility one can discern the source of such an important notion as *polynomial time reducibility* and of studies connected with NP – completeness. Even though Post was not interested in machines Post productions are ubiquitous in computer science and have influenced, for example, Noam Chomsky's work on context-sensitive and context-free languages which provide useful models of languages used in computer-programming. The unsolvability of the Post Correspondence Problem was precisely the requirement to obtain unsolvability results in the theory of formal languages.

In 1936, there was a 'miraculous' confluence of ideas on what mathematically characterises the informal idea of effective calculability. Turing and Post were among the vanguard then and the amazingly rapid development of computer science subsequently has depended much on their creativity and insight. Their place in the history of mathematics is now, deservedly, assured.

Appendix – R.e. sets, the halting problem and the Universal Turing machine.

A set A is r.e. if and only if there is a decidable predicate $R(x, y)$ such that $x \in A$ if and only if $\exists y R(x, y)$ or, *equivalently*, A is either empty or the range of a recursive function.

Now suppose that for some argument x, a Turing machine M_e, encoded by a natural number e, which when applied to an argument x, suitably presented on its tape, computes $f_e(x)$ for some function f_e. Define a T predicate, $T(e, x, y)$, which holds for e, x, y if and only if there is a y which encodes the entire history of one calculation by such a machine from its initial state until it halts. T can be shown to be recursive (actually primitive recursive). We now show that the function $g(x) = f_x(x) + 1$ if $\exists y T(x, x, y)$ and $g(x) = 0$ otherwise is not computable. For suppose $g(x)$ is computed by a Machine M_p, then $g(x) = f_p(x)$ for all x, which gives $g(p) = f_p(p)$ on substituting p for x. Now M_p computes $g(x)$ so we have, for all x, $\exists y T(p, x, y)$ and hence $\exists y T(p, p, y)$. From the definition of $g(x)$ we have $g(p) = f_p(p) + 1$, a contradiction. In fact, the predicate $\exists y T(x, x, y)$ is undecidable (not recursive). For otherwise $g(x)$ would be computed, given x, thus: first decide whether $\exists y T(x, x, y)$; if so, imitate the behaviour of M_x to compute $f_x(x)$ and add 1; if not write 0. This shows that there is a class of quite elementary questions for which there is no decision procedure. The above theorem has come to be known as the *Unsolvability of the halting problem*; there is no effective procedure that, given a machine M and input x, will decide whether or not this calculation ever terminates.

In deference to Post we denote the set $\{x : \exists y T(x, x, y)\}$ by K, so $x \in K$ if and only if $\exists y T(x, x, y)$. Clearly K is r.e. so we conclude that $\bar{K}$ is not r.e.. Post refers to K as a *creative* (complete) set.

Finally, we note that a machine U which computes $f_e(x)$ as a partial function of e and x we call a *Universal Turing machine* since it can be used to compute any computable function $f(x)$. To use it to compute $f(x)$, suppose that $f(x)$ is computed by a machine whose code number is e. U is presented with two numbers e and x as input. It decodes e and proceeds to mimic the machine M applied to x. This shows there are programs, namely *universal programs*, which in a sense incorporate all other programs.

Dedication

This note is dedicated to my late father who, by way of encouraging my scientific studies when I was a sixth former, used to bring home such publications as *Science News* (Penguin Books). One of these led with *Solvable and Unsolvable Problems* by A. M. Turing (February 1954). It was not until my last year at Imperial College, when I attended a course of lectures in the History and Philosophy of Mathematics by G. J. Whitrow, that I became aware of Gödel and incompleteness. Only then did I return to Turing's typically cogent and concise article, which, at last, was clear to me. It was the last of his publications to appear in his lifetime.

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John Sammons, from Southsea, spotted this unusual arithmetic whilst singing carols at the Botley Park Hotel, near Portsmouth.

How many plusses make a minus?

Hayley Mills: It has its plusses and its less plus plusses.
Michael Parkinson: That's a minus.

J H Evans, of Swansea, heard this exchange on Radio 2 on 12th October, 2003.

Summing that series....

In *The Guardian* of 2nd July, 2003, Frederick Forsyth gave this opinion of Tim Henman's chance of winning Wimbledon:-

"...he's only ranked tenth in the world, so I'd say he has a 1 in 10 chance of carrying it off."

Peter Shiu, who noticed this, comments that Forsyth, whilst good at many things, is less good at giving odds, and clearly believes that the harmonic series sums to unity.