The NP-completeness of Vertex Cover

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Basic definitions

- Class NP
  - Set of decision problems that admit "short" and efficiently verifiable solutions
  - Formally, $L \in \mathtt{NP}$ if and only if there exist
    - polynomial p
    - polynomial-time machine V
    - such that, for any x,

$$x \in L \Leftrightarrow \exists y \, (|y| \leq p(|x|) \wedge V(x, y) = 1)$$

- Polynomial-time reducibility
  - $L_1 \leq L_2$ if there exists polynomial-time computable function f such that, for any x,

$$x \in L_1 \Leftrightarrow f(x) \in L_2$$

- $\mathtt{NP}$-complete problem
  - $L \in \mathtt{NP}$ is $\mathtt{NP}$-complete if any language in $\mathtt{NP}$ is polynomial-time reducible to L
    - Hardest problem in $\mathtt{NP}$

Basic results

- Cook-Levin theorem
  - Sat problem
    - Given a boolean formula in conjunctive normal form (disjunction of conjunctions), is the formula satisfiable?
  - Sat is NP-complete
- 3-Sat
  - Each clause contains exactly three literals
- 3-Sat is NP-complete
  - Simple proof by local substitution
    - $l_1 \Rightarrow (l_1 \vee y \vee z) \wedge (l_1 \vee y \vee \overline{z}) \wedge (l_1 \vee \overline{y} \vee z) \wedge (l_1 \vee \overline{y} \vee \overline{z})$
    - $l_1 \vee l_2 \Rightarrow (l_1 \vee l_2 \vee y) \wedge (l_1 \vee l_2 \vee \overline{y})$
    - $l_1 \vee l_2 \vee l_3 \Rightarrow l_1 \vee l_2 \vee l_3$
    - $l_1 \vee l_2 \vee \cdots \vee l_k \Rightarrow$

$$(l_1 \vee l_2 \vee y_1) \wedge (\overline{y_1} \vee l_3 \vee y_2) \wedge (\overline{y_2} \vee l_4 \vee y_3) \wedge \cdots \wedge (\overline{y_{k-3}} \vee l_{k-1} \vee l_k)$$

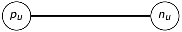
Problem definition: Vertex Cover

Given a graph $G = (N, E)$ and an integer k, does there exist a subset S of at most k vertices in N such that each edge in E is touched by at least one vertex in S?

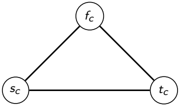
- No polynomial-time algorithm is known
- Is in NP (short and verifiable solution):
  - If a graph is "k-coverable", there exists k-subset $S \subseteq N$ such that each edge is touched by at least one of its vertices
  - Length of S encoding is polynomial in length of G encoding
  - There exists a polynomial-time algorithm that verifies whether S is a valid k-cover
    - Verify that $|S| \leq k$
    - Verify that, for any $(u, v) \in E$, either $u \in S$ or $v \in S$

NP-completeness

- Reduction of 3−Sat to Vertex Cover:
- Technique: component design
  - For each variable a gadget (that is, a sub-graph) representing its truth value
  - For each clause a gadget representing the fact that one of its literals is true
  - Edges connecting the two kinds of gadget
- Gadget for variable u:

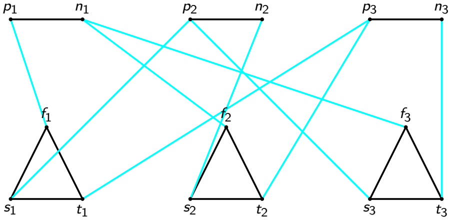


  - One vertex is sufficient and necessary to cover the edge
- Gadget for clause c:



  - Two vertices are sufficient and necessary to cover the three edges
- $k = n + 2m$, where n is number of variables and m is number of clauses

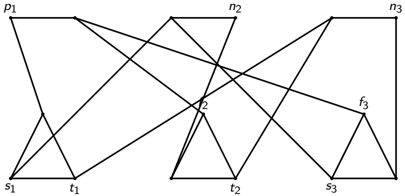
- Connections between variable and clause gadgets
  - First (second, third) vertex of clause gadget connected to vertex corresponding to first (second, third) literal of clause
  - Example: $(x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$



  - Idea: if first (second, third) literal of clause is true (taken), then first (second, third) vertex of clause gadget has not to be taken in order to cover the edges between the gadgets

Proof of correctness

- Show that Formula satisfiable $\Rightarrow$ Vertex cover exists:
  - Include in S all vertices corresponding to true literals
  - For each clause, include in S all vertices of its gadget but the one corresponding to its first true literal
  - Example
    - $(x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3})$
    - x_1 true, x_2 and x_3 false



- Show that Vertex cover exists $\Rightarrow$ Formula satisfiable:
  - Assign value true to variables whose p-vertices are in S
  - Since $k = n + 2m$, for each clause at least one edge connecting its gadget to the variable gadgets is covered by a variable vertex
    - Clause is satisfied