[class 7]

# Limitations of Finite Automata (not in book)

<u>THM</u> (Cantor) There is no bijection $f : S \to 2^S$ if $S$ non-emty.

Proof. Let $S$ be a set and let $f : S \to 2^S$ be a mapping.
We show that $f$ is not a surjection, and hence not a bijection, by giving a counter-example set $\overset{\subseteq S}{ID}$ which we show not to be the image under $f$ of any $x \in S$.
Define $ID \overset{\text{def}}{=} \{x \in S \mid x \notin f(x)\}$ or see below
Now, for any $x \in S$

$$x \in ID \iff x \notin f(x)$$

and therefore

$$ID \neq f(x)$$

$\square$

Presentation of ID by Cantor: inverted diagonal

| | a | b | c | ... |
|---|---|---|---|---|
| f(a) | $\boxed{0}$ | 1 | 0 | |
| f(b) | 1 | $\boxed{1}$ | 0 | |
| f(c) | 0 | 1 | $\boxed{1}$ | |
| ⋮ | | | | |
| ID | 1 | 0 | 0 | |

characteristic functions

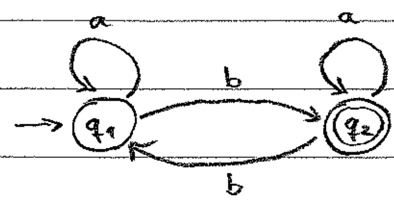
differs from all rows

$\square$

Hence, there is no bijection between $\underbrace{\text{strings over } \Sigma}_{\Sigma^*}$ and $\underbrace{\text{languages over } \Sigma}_{2^{\Sigma^*}}$.

7.2

- Finite automata over $\Sigma$ are injectively encodable as strings over $\Sigma$.

$A$ : 

| | | |
|---|---|---|
| $\hat{A}$ : | aab. | $Q$ |
| | aab. | $\Sigma$ |
| | abaabaabab. | $\delta$ |
| | ab. | $s$ |
| | aabb | $F$ |

Hence the mapping $\hat{A} \mapsto L(A)$ is not a surjection:
There are languages over $\Sigma$ which are not accepted by any FA.
For instance the inverted diagonal language:

$$\{\hat{A} \in \Sigma^* \mid \hat{A} \notin L(A)\}$$

is not a regular language. But also $\{a^n b^n \mid n \geq 0\}$ is not reg.

$\mathcal{A}$

- The argument holds for <u>any</u> class of automata that is injectively encodable in $\Sigma^*$! These are all "traditional" discrete devices.

Terminology: Language $L = \{x \in \Sigma^* \mid P(x)\}$
$P(x)$ - "problem" over $x \in \Sigma^*$
if $L(D) = L$ then $D$ decides problem $P$

Through encoding ^ we can talk about problems over automata (encodings). Self-reference!
Cantor's theorem tells us that there are always "more" problems over $\mathcal{A}$ than automata in $\mathcal{A}$.
No automaton in $\mathcal{A}$ decides the problem $\hat{M} \notin L(M)$.
"Self-reference": if you put $\mathcal{A}$ to decide problems over $\mathcal{A}$
"most" problems are not decidable