DD2422 Exercises 1 - Solutions

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1 Homogeneous coordinates

A line can be written as:

$$ax + by + c = 0 \tag{1}$$

With $l = (a\ b\ c)^T$ and $\overline{x} = (x\ y\ 1)^T$, the equation can be written as $l^T\overline{x}$. Here l is a vector representing the line in the equation, and $\overline{x}$ is the homogeneous description of the 2-D point (x, y).
Note:

- $l^T\overline{x} = \overline{x}^T l = \overline{x} \cdot l$.
- All vectors $\overline{x} = (kx\ ky\ k)^T, k \neq 0$ represents the same point (x, y).
- The vectors not representing a point in $\Re^2$ are on the form $(x\ y\ 0)^T$.
- The vectors not representing a line in $\Re^2$ are on the form $(0\ 0\ c)^T$.

1.1 Intersection of two lines

We want to show that the homogeneous vector we get from the cross product $l \times l_0 = \overline{x}$ represents the intersection between the two lines in the plane.
We have:

$$\overline{x} \text{ lies on } l \ \Leftrightarrow\ l^T\overline{x} = 0 \tag{2}$$

Then

$$l^T\overline{x} = l \cdot \overline{x} = l \cdot (l \times l_0) = 0 \tag{3}$$

since the resulting vector from the cross product of two vectors is perpendicular to both vectors, and the inner product of two perpendicular vectors is zero.
Equivalently:

$$l_0^T\overline{x} = l_0 \cdot \overline{x} = l_0 \cdot (l \times l_0) = 0 \tag{4}$$

Therefore x must lie on both l and l_0, and thus must be the intersection between the lines.

Note:

- If the lines are parallel, i.e. $l = (a\ b\ c)$ and $l_0 = (a\ b\ c_0)$, $c \neq c_0$, we have no solutions, since the resulting vector is on the form $(x\ y\ 0)^T$

1.2 Line from two points

Similar to the previous example, we want to show that the vector we get from the cross product $\overline{x} \times \overline{x}_0 = l$ represents the line that is defined by the two points. We have:

$$l \text{ passes through } \overline{x} \iff l^T \overline{x} = 0. \tag{5}$$

Then

$$l^T \overline{x} = l \cdot \overline{x} = (\overline{x} \times \overline{x}_0) \cdot \overline{x} = 0 \tag{6}$$

(See the reasoning in the previous exercise.)
Equivalently:

$$l^T \overline{x} = l \cdot \overline{x}_0 = (\overline{x} \times \overline{x}_0) \cdot \overline{x}_0 = 0 \tag{7}$$

Therefore l must pass through both $\overline{x}$ and $\overline{x}_0$, and thus the both points define the line.

Note:

- Unlike part 1.1, the cross product between two vectors with only the third element differing, represents a line. In this case the two points lie in the same direction from the origin, but at different distances.

2 Projective transformations

We have

$$y = Ax \tag{8}$$

and since A is non-singular

$$x = A^{-1} y \tag{9}$$

2.1 Line

The line in the first plane is

$$l_a^T x = 0 \tag{10}$$

Equation 9 gives

$$l_a^T A^{-1} y = 0 \tag{11}$$

Rearranging, and using the notation A^{-T} for the transpose of the inverse of A, gives

$$\left(A^{-T} l_a\right)^T y = 0 \tag{12}$$

Which gives the line in the second plane

$$l_b^T y = 0 \tag{13}$$

where

$$l_b = A^{-T} l_a \tag{14}$$

2.2 Ellipse

The same method can be applied to an ellipse. The ellipse in the first plane is

$$x^T C_a x = 0 \tag{15}$$

Equation 9 gives

$$y^T A^{-T} C_a A^{-1} y = 0 \tag{16}$$

Which gives the ellipse in the second plane

$$y^T C_b y = 0 \tag{17}$$

where

$$C_b = A^{-T} C_a A^{-1} \tag{18}$$

3 Histogram normalisation

We have

$$p(z) = \frac{\pi}{2z_{\mathrm{max}}} \sin\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right) \tag{19}$$

The transfer function will be of the form

$$s = T(z) = c_0 + c_1 \int_0^z p(z)\,\mathrm{d}z \tag{20}$$

where

$$T(0) = -\frac{z_{\mathrm{max}}}{2} \qquad \text{and} \qquad T(z_{\mathrm{max}}) = \frac{z_{\mathrm{max}}}{2} \tag{21}$$

That gives

$$T(0) = c_0 + c_1 \int_0^0 p(z)\,\mathrm{d}z = c_0 \Leftrightarrow c_0 = -\frac{z_{\mathrm{max}}}{2} \tag{22}$$

and

$$\begin{aligned}
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1 \int_0^{z_{\mathrm{max}}} p(z)\,\mathrm{d}z && (23)\\
&\Leftrightarrow \\
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1 \frac{\pi}{2z_{\mathrm{max}}} \int_0^{z_{\mathrm{max}}} \sin\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right) \mathrm{d}z && (24)\\
&\Leftrightarrow \\
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1 \frac{\pi}{2z_{\mathrm{max}}} \frac{-2z_{\mathrm{max}}}{\pi} \left[\cos\left(\frac{\pi}{2}\frac{z}{z_{\mathrm{max}}}\right)\right]_0^{z_{\mathrm{max}}} && (25)\\
&\Leftrightarrow \\
T(z_{\mathrm{max}}) &= -\frac{z_{\mathrm{max}}}{2} + c_1(-1)(0-1) && (26)\\
&\Leftrightarrow \\
c_1 &= T(z_{\mathrm{max}}) + \frac{z_{\mathrm{max}}}{2} = z_{\mathrm{max}} && (27)
\end{aligned}$$

The transfer function is therefore

$$
\begin{aligned}
T(z') &= -\frac{z_{\max}}{2} + z_{\max} \int_0^{z'} p(z)\,\mathrm{d}z && (28)\\
&\Leftrightarrow \\
T(z') &= -\frac{z_{\max}}{2} + z_{\max} \frac{\pi}{2 z_{\max}} \int_0^{z'} \sin\left(\frac{\pi}{2}\frac{z}{z_{\max}}\right) \mathrm{d}z && (29)\\
&\Leftrightarrow \\
T(z') &= -\frac{z_{\max}}{2} + \frac{\pi}{2} \int_0^{z'} \sin\left(\frac{\pi}{2}\frac{z}{z_{\max}}\right) \mathrm{d}z && (30)
\end{aligned}
$$

Stretching of grey level values is defined by

$$
\frac{\partial T}{\partial z} > 1 \tag{31}
$$

in the case where the span of the original intensities is equal to the span of the transformed intensities. In this case the span is the same and equal to $z_{\max}$. What would it look like in the general case?

$$
\begin{aligned}
\frac{\partial T}{\partial z} &> 1 && (32)\\
&\Leftrightarrow \\
z_{\max}\, p(z) &> 1 && (33)\\
&\Leftrightarrow \\
\frac{\pi}{2}\sin\left(\frac{\pi}{2}\frac{z}{z_{\max}}\right) &> 1 && (34)\\
&\Leftrightarrow \\
\sin\left(\frac{\pi}{2}\frac{z}{z_{\max}}\right) &> \frac{2}{\pi} && (35)\\
&\Leftrightarrow \\
\frac{\pi}{2}\frac{z}{z_{\max}} &> \arcsin\left(\frac{2}{\pi}\right) \quad \text{when} \quad 0 \le z \le z_{\max} && (36)\\
&\Leftrightarrow \\
z &> z_{\max}\frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\right) && (37)
\end{aligned}
$$

That is, stretching of the grey level values occur when $z > z_{\max}\frac{2}{\pi}\arcsin\left(\frac{2}{\pi}\right)$.

4 Convolution

$$
\begin{aligned}
F &= \{0, 1, 2, 3, 11, 4, 0\} && (38)\\
G_1 &= \{1, 2, 4\} && (39)\\
G_2 &= \{1, 0, -1\} && (40)\\
G_3 &= \{1, -2, 1\} && (41)
\end{aligned}
$$

$$
f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y \tag{42}
$$

$$f(x) = \begin{cases} F(x) & \text{if } 1 \leq x \leq 7 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{43}$$

$$g(x) = \begin{cases} G(2+x) & \text{if } -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{44}$$

$$f * g(x) = \begin{cases} F * G(x-1) & \text{if } 2 \leq x \leq 6 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{45}$$

$$F * G(x-1) = \begin{cases} \sum_{y=-1}^{1} F(x-y)G(2+y) & \text{if } 2 \leq x \leq 6 \\ \text{undefined} & \text{otherwise} \end{cases} \tag{46}$$

$$F * G_1 = \{4, 11, 25, 38, 52\} \tag{47}$$

$$F * G_2 = \{2, 2, 9, 1, -11\} \tag{48}$$

$$F * G_3 = \{0, 0, 7, -15, 3\} \tag{49}$$

5 Discrete Fourier transform

Whe have

$$v(0 \ldots 3) = 1, 2, 3, 5 \tag{50}$$

The discrete Fourier transform is

$$\hat{v}(m) = \frac{1}{N} \sum_{n=0}^{N-1} v(n) e^{-i \, \frac{n}{N} m \, 2\pi} \tag{51}$$

This gives

$$\hat{v}(0) = \frac{1}{4}(1 + 2 + 3 + 5) \tag{52}$$

$$\hat{v}(1) = \frac{1}{4}(1(1) + 2(-i) + 3(-1) + 5(i)) \tag{53}$$

$$\hat{v}(2) = \frac{1}{4}(1(1) + 2(-1) + 3(1) + 5(-1)) \tag{54}$$

$$\hat{v}(3) = \frac{1}{4}(1(1) + 2(+1) + 3(-1) + 5(-i)) \tag{55}$$

$$\Leftrightarrow$$

$$\hat{v} = \frac{1}{4}(11,\ -2 + 3i,\ -3,\ -2 - 3i) \tag{56}$$

6 Mirroring

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y) g(y) dy \tag{57}$$

$$f_- * g_-(x) = \int_{-\infty}^{\infty} f_-(x-y) g_-(y) dy \tag{58}$$

$$f_- * g_-(x) = \int_{-\infty}^{\infty} f(-x-(-y)) g(-y) dy \tag{59}$$

$$f_- * g_-(x) = \int_{\infty}^{-\infty} f(-x - y^*)g(y^*)(-dy^*) \text{ where } y^* = -y \quad (60)$$

$$f_- * g_-(x) = \int_{-\infty}^{\infty} f(-x - y^*)g(y^*)dy^* = f * g(-x) \quad (61)$$

$$f_- * g_- = (f * g)_- \quad (62)$$

7 Continuous Fourier transform

We have in spatial space

$$f(x) = \begin{cases} 1 + x & \text{if} \quad -1 < x < 0 \\ 1 - x & \text{if} \quad 0 \leq x < 1 \\ 0 & \text{otherwise} \end{cases} \quad (63)$$

This function is the convolution of two box functions

$$f(x) = g * g(x) \quad (64)$$

where

$$g(x) = \theta(x + \frac{1}{2}) - \theta(x - \frac{1}{2}) \quad (65)$$

Here $\theta(x)$ is the Heaviside step function, and is defined as

$$\theta(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ 0 & \text{if} \quad x < 0 \end{cases} \quad (66)$$

For $x = 0$ it is defined differently depending on the application. In this case, it can be defined as $\theta(0) = 1$.
The Fourier transform of g is (see Beta p. 319, F50)

$$\hat{g}(w) = \frac{2}{w} \sin\left(\frac{1}{2}w\right) \quad (67)$$

That together with equation 64 and the Fourier transform of a convolution (Beta p. 317, F13) gives

$$\hat{f}(w) = \boldsymbol{F}(f)(w) = \boldsymbol{F}(g * g)(w) = \boldsymbol{F}(g)\boldsymbol{F}(g) = \hat{g}^2(w) = \frac{4}{w^2} \sin^2\left(\frac{w}{2}\right) \quad (68)$$

This is what the function looks like: