DD2440 Lecture 11

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1 TSP and Held-Karp Integrality Gap

The heuristic for linear programming for hard problems. Relax $x \in \{0,1\}$ to $x_i \in [0,1]$

$$x_{ij} = \begin{cases} 1 & \text{an edge in tour between i and j} \\ 0 & \text{otherwise} \end{cases}$$

Held-Karp:

$$\min \sum d_{ij} x_{ij}$$

$$\sum_{j \neq i} x_{ij} = 2 \quad \forall i$$

$$\sum_{\substack{i \in S \\ j \notin S}} x_{ij} \geq 2 \text{ (*)}$$

We want to make an integer linear programming (ILP) with exponential number of constants into an LP with few constraints. LP: $x_{ij} \in [0,1]$ By being clever you can write down a few constraints that imply (*). This was not done in the lecture but let us sketch how to do this. We want to say that for any nodes s and t there is a unit flow from s to t using x_{ij} as capacities. This is coded by variables y_{ijst} which is supposed to give the flow from vertex i to vertex j in this flow. We have the following constraints.

1. $y_{isst} = 0$ for all i. No flow into s.
2. $\sum_j y_{sjst} = 1$. Flow one out of s.
3. $y_{tjst} = 0$ for all j. No flow out of t.
4. $\sum_i y_{itst} = 1$. Flow one into t.
5. $\sum_i y_{ikst} = \sum_j y_{kjst}$ for $k \neq s, t$. Flow conservation at other nodes.
6. $0 \leq y_{ijst} \leq x_{ij}$. The x_{ij} are capacities.

This gives a polynomial size LP and implies the constraints (*).

The benefits of an LP-relaxation for a hard problem:

Always give you an idea of the optimal value. LP has at least as good optimum as ILP and we can always solve the LP efficiently.

Sometimes we can use LP solution to find a good integer solution.

Key Question: How good is this estimate?

Key Notation: Integrality gap. Instances with good fractional solution. Bad integral solution.

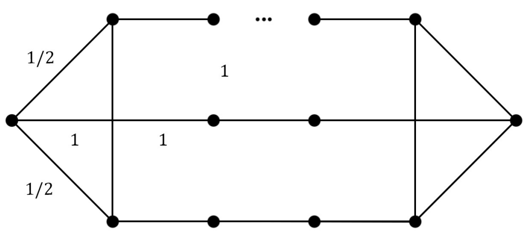

Figure 1: Three parallel lines of edges with cost one. Triangles at the end also with edges of length one hook up the end-points.

A different way of picturing the same graph is seen below. This is done by drawing one triangle large and the other, small edges, are still of cost one.

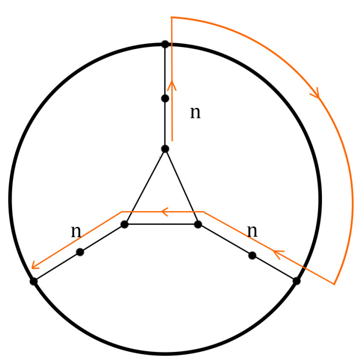

Figure 2: Cost of outer edges is 1

An optimal tour needs to return to the start. In Figure 2, we need to traverse one of the long paths between the inner triangle and the outer triangle twice to get to the starting point. The length of the tour is thus roughly $4n$ since the cost of the outer edges can be neglected. A fractional solution, with length 1/2 on triangles edges, 1 on all other edges as can be seen in Figure 1 has the cost about $3n$. We get the following theorem.

Theorem 1. *The integrality gap for Held-Karp relaxation is at least* $4/3$.

The upper bound for the integrality gap is more complicated and we only state the theorem.

Theorem 2. *Held-Karp is within a factor* $3/2$.

The Held-Karp relaxation, after a minor modification, works also for asymmetric TSP, i.e. where $d_{ij} \neq d_{ji}$ might be true, but we do have triangle inequality. For ATSP Held-Karp can be proved to be within a factor of $O(\frac{log(n)}{log(log(n))})$ of OPT while the worst instances has a gap of a factor of 2. Which of the two bounds that is the better is a question for future research.

As for algorithm independent hardness results it is known that TSP is NP-hard to approximate within 123/122 while the bound for ATSP is 75/74. Help-Karp may be the best algorithm, but this remains to be seen.

2 Max-Cut

The Max-Cut problem is to find the cut which cuts the maximum number of edges by cutting a graph into two pieces.

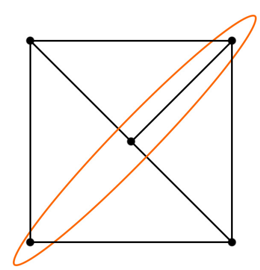

Figure 3: Maximum cut

To tell if it is possible to cut all the edges in a graph is the same as determining whether the graph is bipartite or not, and this is easy to tell. By using random cuts we cut half of the edges in the graph, we want to do better than that. We want to maximize the number of cut edges and this number is captured by the following expression

$$\max \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}, \quad x_i \in \pm 1.$$

We relax this to

$$max \sum_{i,j \in E} \frac{1 - y_{ij}}{2},$$

where y is a symmetric, positive, semidefinite matrix with $y_{ii} = 1$, i.e.

$$y_{i,j} = \begin{pmatrix} 1 & ? & \cdots & ? \\ ? & 1 & \cdots & ? \\ \vdots & \vdots & \ddots & \vdots \\ ? & ? & \cdots & 1 \end{pmatrix}$$

Solving this will give an 0.878 approximation for Max-Cut and let us give some details.

Recall properties of a symmetric, positive, semidefinite matrix. What do the eigenvalues of symmetric matrix look like?

Symmetric: Real eigenvalues

Positive, semi definite: $\lambda_i \geq 0$ for $i = 0, 1, \cdots, n-1$.

The following conditions are equivalent:

E1: $\sum_{ij} y_{ij} z_i z_j \geq 0 \; \forall \; z \in \mathbb{R}^n$.

E2: $y = V^T V$ for some matrix V.

If $y_{ij} = x_i x_j$ for some $x \in \{\pm 1\}^n$ is this PSD? Looking at E2 we realize that $Y = (x^T 0)\binom{x}{0}$ and thus we know that we have a relaxation.

Is the relaxation solvable in polynomial time? (This is previously why we used LP). The answer is yes. Let us explain the meta reason for this. Remember for previous lectures, why is LP easy?

- No local maximas
- $Ax \leq b$ (nice convex domain)

Could we reuse this idea? We need to check that the set of positive semidefinite matrices is a convex domain and in particular that if Y^1 is PSD and Y^2 is PSD then so is $\lambda Y^1 + (1-\lambda)Y^2$ for any $\lambda \in [0,1]$. This is easy to see by condition E1 as if

$$z^T Y^1 z \geq 0 \quad z^T Y^2 z \geq 0$$

both hold then

$$z^T(\lambda Y^1 + (1-\lambda)Y^2)z = \lambda z^T Y^1 z + (1-\lambda) z^T Y^2 z \geq 0.$$

If $V_i \in R^n$ is the i'th column of V in $Y = V^T V$ then $Y_{ij} = (V_i, V_j)$ (the inner product) and $y_{ii} = 1$ implies $\|V_i\| = 1$. Thus one way to formulate the relaxation is that we have gone from ± 1 (unit vectors in $\mathbb{R}^1$) to unit vectors in $\mathbb{R}^n$.

We need to turn this vector solution back into a Boolean solution. To do this we pick a random unit vector $r \in \mathbb{R}^n$. Set $x_i = sign((v_i, r))$. In the picture we see the situation when the vectors all happen to lie in the same two-dimensional space.

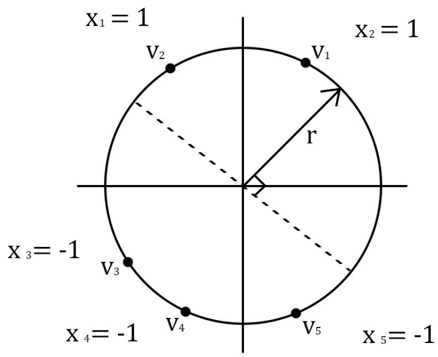

Figure 4: The vectors v_i in two dimensions.

Intuition: If $\frac{1-(v_i,v_j)}{2}$ gives a large contribution to objective function then v_i, v_j point opposite and it is likely that $x_i \neq x_j$. Let us try to make this formal.

OPT: Best Max-Cut. SDP-OPT: Optimal value of SDP.

1. SDP-OPT $\geq$ OPT (it is a relaxation)
2. Expected value of constructed solution is $\geq$ 0.878. SDP-OPT $\geq$ 0.878 OPT

Let us sketch why this the case. Suppose we have an edge (i, j) and the angle between V_i and V_j is θ. Then $\frac{1-(v_i,v_j)}{2} = \frac{1-cos\theta}{2}$. It is not difficult to see that

$$Prob[x_i \neq x_j] = \frac{\theta}{\pi}$$

and as

$$\min_{\theta} \frac{\frac{\theta}{\pi}}{\frac{1-cos(\theta)}{2}} \approx 0.878$$

this gives the desired conclusion. This was done rather quickly and will be discussed again at the beginning of next lecture.