Transcript of lecture 1
Computational model and sorting

2014-09-24, 14:15-15:00

Notes by Jonatan Asketorp (asket@kth.se).

Random Access Machine

Unit-cost RAM is the main model used in this course. This is similar to C or any other standard imperative language.

- Any operation costs one unit.
- Indirect addressing available.

A variation is the log cost RAM where an operation on an integer x costs $\log(1 + x)$, this is more accurate but more tedious.

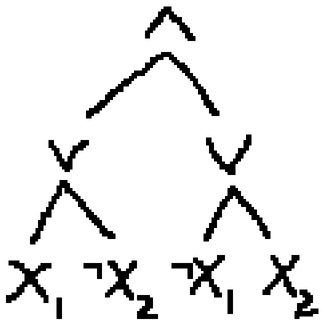
Interesting model: Boolean circuit

Boolean circuits are bit oriented with input bits, $x_1, ..., x_n$ and operands

$$\begin{aligned} \wedge &= \text{"and"} \\ \vee &= \text{"or"} \\ \neg &= \text{"negation"} \end{aligned}$$

The size is measured as the number of logic gates.

Example: XOR



which has size 3.

Turing machines

Will not be used in this course.

Interesting model: Comparison model

A common claim is that it requires $n\log(n)$ time to sort n integers. However, this is only true in the comparison model where the only way to get information about the input is asking "Is $x_i \geq x_j$?"

Theorem 0.1 *Sorting requires* $\log_2(n!)$ *comparisons in this model where* $!$ *is the factorial function* $n! = n * (n-1) * (n-2) * ... * 1$.

Proof: The output is an ordering $l_1, l_2, l_3, ..., l_n$ (i.e, a permutation of n) where $x_{i_1} \geq x_{i_2} \geq x_{i_3} \geq ... \geq x_{i_n}$.

There are $n!$ possible answers and the correct answer is found by using information about the input. If we ask T questions "$x_i \geq x_j$?" we have 2^T possible answer sets and each answer set gives an output. Note that each output is possible!

The number of answer sets is $\geq$ the number of outputs which means that $2^T \geq n!$ and $T \geq \log_2(n!)$. ∎

An approximation of $n!$ by Stirling's formula is $n! \sim (\frac{n}{e})^n$. This gives us that $\log_2(n!) \sim n\log_2 n - n\log_2 e + o(n)$.

Sorting algorithms

Quicksort is average case $O(n \log n)$ but it gives the wrong leading constant. Mergesort and Heapsort, are sorting algorithms that gives the optimal constant one in front of $n \log n$.

Sort n numbers with 64 bits each.

How long does it takes to sort these?

Proposed algorithm:

- Initialize 2^{64} counters to 0.
- for $i = 1$ to n increase counter C_{x_i}.
- Read off answer.

Time $n + O(1)$. An objection to this is that in real life n is between 2^{20} and 2^{45} as smaller n are "easy" and larger n we cannot even read the input. This algorithm is "cheating" as the $O(1)$ is the dominating term.

Radix sort

Sorts n numbers in range $0...n^k - 1$ in time $\sim kn$.

Bucket sort

Bucket sort is a similar algorithm.

- Make n buckets, say $n = 2^{22}$.
- Put elements in bins given by the first $\log n$ bits (22 bit).
- Sort bins recursively, now with 42 bit numbers.

There will be 3 levels of recursion and the time in n is $3n$ + book-keeping.

Is sorting in reality O(n) time?

Think about what is the best/worst combination of n and w (the number of bits in the numbers)? For $w \leq 3 \log n$ sorting can, as discussed above, be done in linear time! On the other hand if w is very large then each comparison may take a very long time. One model that strikes a reasonable balance is the following. We want to sort n numbers each with w bits and we allow simple machine operations of w-bit numbers at unit cost. The current world record in this model is an algorithm that sorts in $O(n \log \log n)$ time by A. Andersson, T. Hagerup, S. Nilsson and R. Raman. A link to this is available on the homepage. It currently is unknown whether it can be done in $O(n)$!

Circuit model sorting

A circuit model of n w-bit numbers with $m = wn$ input bits. Can sorting be done in $O(m)$ size? This is also unknown.

What we did not have time for

Given n random integers each w bit, i.e. $x_i \in 0...2^w - 1$ randomly. It is easy to sort in $O(n)$ time. See bucket sort with n buckets. Let s_x be the $\log n$ most significant bits in x and simply place x in B_{s_x} and sort the buckets by almost any method (even a with a quadratic sorting algorithm for the buckets, this can be proved to run in expected time $O(n)$). The proof was skipped but a sketch is a available in the course notes in Section 18.5.