DD2440: Advanced Algorithms **Fall 2014**

Lecture 5: October 10

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5.1 The Chinese remainder theorem (CRT)

With $N = pq$ and $GCD(p,q) = 1$, then

$$x = \begin{cases} x_1 & \text{mod p} \\ x_2 & \text{mod q} \end{cases}$$

has a unique solution modulo N that can be found efficiently (polynomial time with regard to $\log N$).

Efficiently solve it for the special cases

$$\begin{aligned} (x_1, x_2) = (1,0) \quad &\Rightarrow \text{ solution } u_1 \\ (x_1, x_2) = (1,0) \quad &\Rightarrow \text{ solution } u_2 \end{aligned}$$

We can then calculate a general solution x

$$x = x_1 u_1 + x_2 u_2 \text{ mod N}$$

since

$$\begin{aligned} x = x_1 u_1 + x_2 u_2 = x_1 \cdot 1 + x_2 \cdot 0 \text{ mod p} \\ x = x_1 u_1 + x_2 u_2 = x_1 \cdot 0 + x_2 \cdot 1 \text{ mod q} \end{aligned}$$

We can compute u_1 and u_2 by running the extended Euclidean algorithm on p and q. We get a and b such that

$$1 = \underbrace{ap}_{u_1} + \underbrace{bq}_{u_2}$$

since

$$\begin{aligned} 1 = ap + bq \text{ mod p} = 0 + bq \text{ mod p} \Rightarrow bq = 1 \text{ mod p} \\ 1 = ap + bq \text{ mod q} = ap + 0 \text{ mod q} \Rightarrow ap = 1 \text{ mod q} \end{aligned}$$

5.2 Modular division

What is $\frac{2}{3}$ mod 7 ?

$$\begin{aligned} 3 \cdot \tfrac{2}{3} = 2 \text{ mod 7} \\ 3 \cdot x = 2 \text{ mod 7} \end{aligned}$$

We see that $x = 3$ does it!

What is $\frac{2}{3}$ mod 6 ?

$$3 \cdot x = 2 \text{ mod 6}$$

This has no solution and this should not be a surprise. Already in the real numbers we know that $\frac{2}{0}$ is not defined. It is bad with a zero in the denominator. The Chinese remainder theorem states that modulo 6 is the same as modulo 2 and modulo 3 at the same time and $\frac{2}{3}$ modulo 6 when we look at it modulo 3, this is $\frac{2}{0}$.

5.3 Efficient modular division

Think of $\frac{2}{3}$ as $2 \cdot \frac{1}{3}$. How do we compute modular inverses?

Use the extended Euclidean algorithm

$$GCD(p, b) = 1 \Rightarrow 1 = cp + db \Rightarrow d = \frac{1}{b}$$

5.4 Factorization

We want to factor $N = p \cdot q$ in less than linear time with regard to p (which is the time complexity for trial division).

5.4.1 Pollard's ρ algorithm - magic and simple algorithm

Algorithm:

$$\begin{aligned} &x_0 = 4711 \\ &x_{i+1} = x_i^2 + 1 \text{ mod N} \end{aligned}$$

Compute $GCD(x_{2i} - x_i, N)$ for $i = 1, 2, ...$ until you find a factor ($GCD(x_{2i} - x_i, N) \neq 1$, the value of GCD(...) is a factor of N)

CRT: modulo $N \sim$ modulo p & modulo q. Squaring is pretty random; x_i modulo p looks like random numbers until we get a repeat.

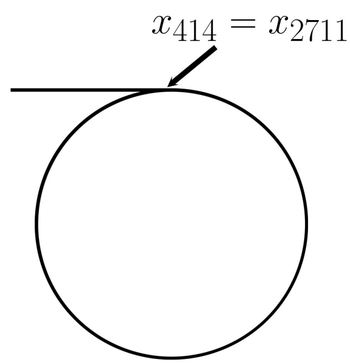
$$\begin{aligned} &x_0 = 4711 \\ &x_1 = \text{ Some number mod p} \\ &... \\ &x_{2711} = x_{414} \\ &x_{2712} = x_{415} \end{aligned}$$

Iterating x_i mod p is equivalent to running around a loop. x_{2i} is running twice as fast as x_i. When they meet up $x_{2i} - x_i$ is divisible by p and $GCD(x_{2i} - x_i, N)$ contains the factor p (we are not sure that this is a prime but that can easily be checked).

Heuristic statement: Pollard ρ finds the factor p in $\sim \sqrt{p}$ time. This is based on the heuristic assumption that the x_i behave like random numbers and the key is to analyze how many random numbers are needed until we get a repeated value.

5.4.2 Collision probability

How many random numbers ($x_i's$ mod p) are needed to get a repeat?



This is analogous to the birthday problem (what's the probability of collision of birthdays in a group of a certain size?). The probability of no collision is $\sim e^{\frac{-t^2}{2p}}$, $t = \#numbers$ and p is the nunber of possibilities..

$$\begin{array}{ll} t \sim \sqrt{2p} & \rightarrow P(\text{no collision}) = e^{-1} \\ t \sim \sqrt{10p} & \rightarrow P(\text{no collision}) = e^{-5} \end{array}$$

5.4.3 Implementation

```
x = 4711
y = 4711
repeat
        x = x^2 + 1 mod N
        y = y^2 + 1 mod N
        y = y^2 + 1 mod B
        if (GCD(x-y,N) != 1) return GCD(x-y,N)
```

The squaring and modulo calculations are considerably faster than the GCD calculation, thus we want to perform few calls to GCD. We can achieve this by multiplying together a few consecutive $x_{2i} - x_i$ mod N before calling GCD on the product.

5.4.4 General factorization

Find nontrivial solution ($x \neq \pm y$) to $x^2 = y^2$ mod N.

N divides $x^2 - y^2 = (x-y)(x+y)$ but not either factor. $GCD(N, x-y)$ is a factor of N.

First idea: Small numbers are often squares, $\lceil\sqrt{N}\rceil$ ($\lceil \quad \rceil$ means round up to next integer).

Example:

$$\begin{array}{l} N = 21 \\ \lceil\sqrt{21}\rceil = 5 \\ 5^2 = 25 = 4 = 2^2 \text{mod } 21 \\ GCD(5-2, 21) = 3 \\ GCD(5+2, 21) = 7 \end{array}$$

How large is $\lceil\sqrt{N}\rceil^2 - N$?

$$\begin{aligned}&\lceil\sqrt{N}\rceil - \sqrt{N} \sim \tfrac{1}{2}\\ &\tfrac{d\sqrt{N}^2}{d\sqrt{N}} = 2\sqrt{N}\\ &\Rightarrow \lceil\sqrt{N}\rceil^2 - N \approx \sqrt{N}^2 + \tfrac{1}{2} * 2\sqrt{N} - N = \sqrt{N}\end{aligned}$$

In the last step we assume that the ceiling operation adds an average of $\frac{1}{2}$ to $\sqrt{N}$ and substitute $\lceil\sqrt{N}\rceil^2$ with a Taylor expansion.

What is the probability that a number of size T is a perfect square?

There are $\lfloor\sqrt{T}\rfloor$ perfect squares $\leq T$, which means that the probability is $\sim \frac{1}{\sqrt{T}}$.

In our case we have $T \sim \sqrt{N}$, which gives us a probability of $\approx N^{-\frac{1}{4}}$ that $\sqrt{N}$ is a square, and a time complexity of $N^{\frac{1}{4}} \geq \sqrt{p}$ where p is the smallest prime and thus Pollard's ρ algorithm is better.

Example:

$$\begin{aligned}&N = 161\\ &\lceil\sqrt{161}\rceil = 13\\ &13^2 = 169 = 8 \bmod 161 \text{ (not a square)}\\ &\lceil\sqrt{2 * 161}\rceil = 18\\ &18^2 = 324 = 2 \bmod 161 \text{ (not a square)}\\ &13^2 \cdot 18^2 = (13 \cdot 18)^2 = 8 \cdot 2 = 4^2\\ &13 \cdot 18 \bmod 161 = 73\\ &GCD(73-4, 161) = 7\\ &GCD(73+4, 161) = 13\end{aligned}$$

Example:

$$\begin{array}{ll} N = 123 & \\ 11^2 = 121 = -2 & \bmod 123\\ 12^2 = 144 = 21 = 3 \cdot 7 & \bmod 123\\ 16^2 = 256 = 10 = 2 \cdot 5 & \bmod 123\\ 18^2 = 324 = -45 = -5 \cdot 3^2 & \bmod 123\\ 19^2 = 361 = -8 = -2^3 & \bmod 123\end{array}$$

We can find squares by combining the above

$$\begin{array}{ll} (11 \cdot 19)^2 = -2 \cdot -2^3 = 2^4 = 4^2 & \bmod 123\\ (11 \cdot 16 \cdot 18)^2 = -2 \cdot 2 \cdot 5 \cdot -5 \cdot 3^2 = (2 \cdot 3 \cdot 5)^2 & \bmod 123\end{array}$$

5.4.5 Quadric Sieve

Idea: Generate many ($\sim 10^6$) a_i such that $b_i = a_i^2$ are small mod N. One good alternative is to use.

$$b_i = (i + \lceil\sqrt{N}\rceil)^2 - N$$

Factor all b_i and combine to form perfect squares. More about this in next lecture.