DD2441 LECTURE 2

2-player deterministic comm cplx

Alice gets $x \in X$

Bob gets $y \in Y$

Need to communicate (send bits back and forth) to compute $f(x,y)$ $\quad f: X \times Y \to Z$

Cost of protocol P = # bits sent on worst-case (x,y)

Communication complexity of f = cost of best protocol P that computes f. [$D(f)$ or $CC_D(f)$]

Last time: Upper bounds (i.e., concrete protocols)

E.g. $\quad D(\text{MED}) = O(\log n)$ $\quad$ [Ex 5]

$\quad\quad D(\text{EQ}) \leq n+1$ $\quad$ [Ex 7]

Today: Lower bounds

$$\text{Range}(f) = \{ z \in Z \mid \exists (x,y) \text{ s.t. } f(x,y) = z \}$$

PROP 8 $\quad D(f) \geq \log_2 |\text{Range}(f)|$

<u>Proof</u> Recall can describe protocol as binary tree.
At termination, answer known to both A & B
So leaves labelled by $z \in Z$.
At least $m = |\text{Range}(f)|$ distinct leaves.
$\Rightarrow$ height of binary tree $\geq \log_2 m$

COR 9 $D(\text{MOD}) = \Theta(\log n)$

Proved $O(\log n)$ bound last time.
And n possible answers.

<u>But</u> comm cplx mostly focuses on Boolean functions $Z = \{0,1\}$
Then Prop 8 useless...

[Non-Boolean case also interesting, but most techniques extend from Boolean to non-Boolean case.]

In Boolean case, won't ~~insist~~ always that output is clear from protocol — just changes comm cplx by additive 1 anyway.

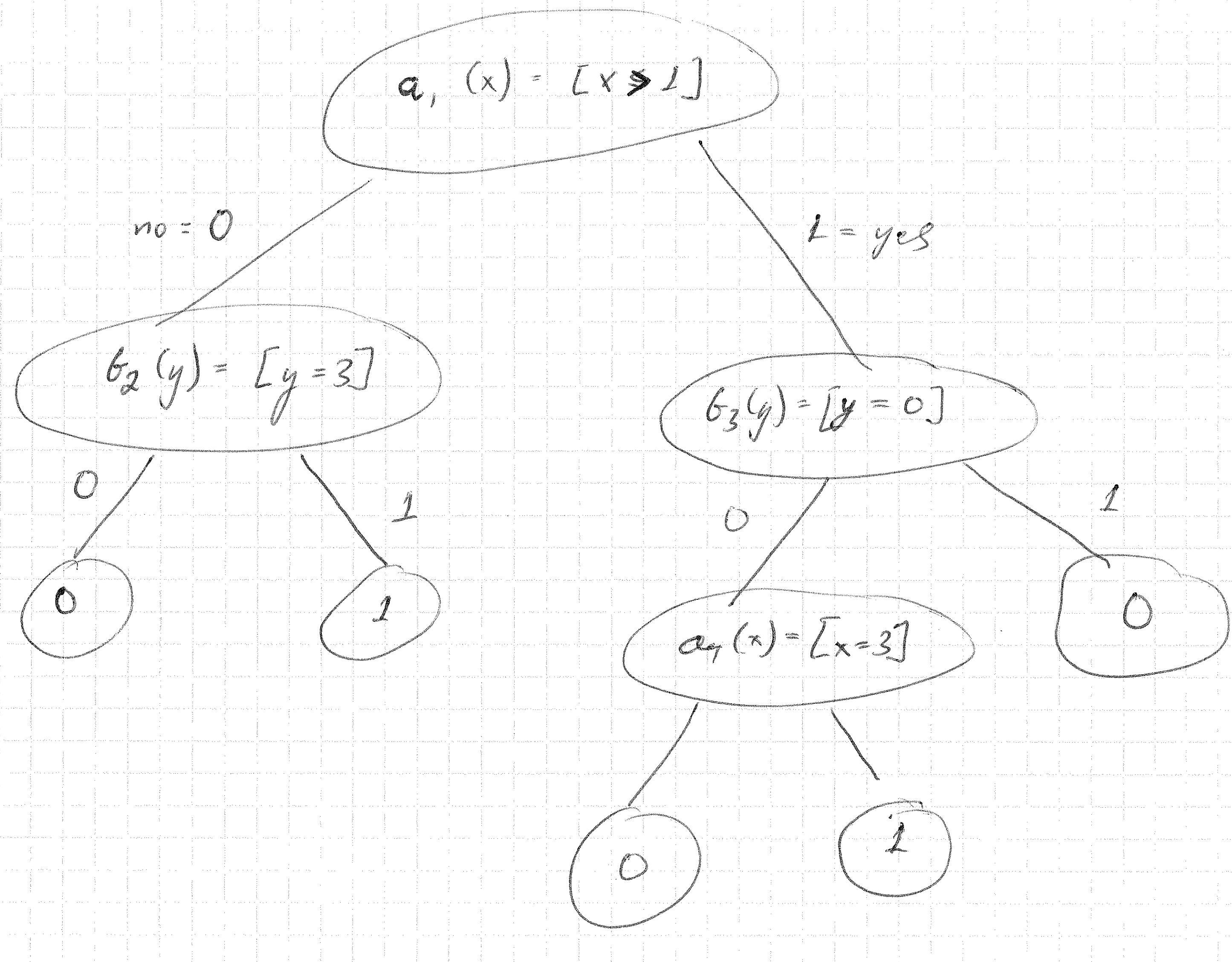
To prove LB, view protocol as a way of partitioning $X \times Y$ into sets so that for each set same communication is sent. Then analyze structure of such sets and partitions.

To illustrate, give function and protocol on next page.

Ex 10

$x, y \in \{0, 1, 2, 3\}$

If $x \leq 1$ then $[y = 3]$
else if $x = 2$ then 0
else $[y \neq 0]$



Function table

x \ y	00	01	10	11
00	0	0	0	1
01	0	0	0	1
10	0	0	0	0
11	0	1	1	1

For P protocol, v node in protocol tree,
R_v is set of inputs (x,y) reaching v.

If $\mathcal{L}$ = leaves of protocol P, then
$\{R_\ell\}_{\ell \in \mathcal{L}}$ is partition of $\mathcal{X} \times Y$

<u>DEF 11</u> A (combinatorial) rectangle in $X \times Y$ is a subset $R \subseteq X \times Y$ s.t. $R = A \times B$ for some $A \subseteq X$, $B \subseteq Y$.

<u>PROP 12</u> $R \subseteq \mathcal{X} \times Y$ is a rectangle iff
$(x_1, y_1) \in R$ and $(x_2, y_2) \in R \Rightarrow (x_1, y_2) \in R$

<u>Proof</u> Exercise.

<u>PROP 13</u> For every P and every protocol node v, R_v is a rectangle (in particular holds for leaves)

<u>Proof</u> Use Prop 12. Suppose $(x_1, y_1) \in R_v$ $(x_2, y_2) \in R_v$, and show $(x_1, y_2) \in R_v$

By induction. True for root of protocol tree.

Suppose w child of v, $(x_1, y_1), (x_2, y_2) \in R_w$

Then by IH $(x_1, y_2) \in R_v$. If Alice speaks at v, she only sees transcript and x_1, and so will go to w for both (x_1, y_1) and (x_1, y_2)

If Bob speaks transcripts look the same for (x_1, y_2) and (x_2, y_2) so Bob moves to same node on both.

Inductively, every time Alice speaks scenarios for (x_1, y_1) and (x_1, y_2) are indistinguishable for her, and similarly for Bob. ∎

$R \subseteq X \times Y$ is monochromatic iff f fixed on R.

If ℓ leaf of protocol, then R_ℓ is monochromatic (by definition). Hence:

LEMMA 14 Any protocol $\mathcal{P}$ for function $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ induces a partition of $\mathcal{X} \times \mathcal{Y}$ into monochromatic rectangles
[# rectangles = # leaves in protocol tree]

COR 15 If any partition of $\mathcal{X} \times \mathcal{Y}$ into monochromatic rectangles requires $\geq t$ rectangles, then $D(f) \geq \log_2 t$

Proof Same as for Prop 8.
Need $\geq t$ leaves $\Rightarrow$ height $\geq \log_2 t$. ∎

Look at Ex 10 again

	00	01	10	11
11	0	0	0	1
01	0	0	0	1
10	0	0	0	0
11	0	1	1	1

Note that combinatorial rectangles need not be geometric rectangles (although convenient to draw this way in examples.

FOOLING SETS [Yao '79, Lipton & Sedgewick '81]

Exhibit large set of input pairs s.t.
no two pairs (x,y) (x',y') can be in
same monochromatic rectangle
$\Longrightarrow$ # monochromatic rectangles large
$\Longrightarrow$ CC lower bound.

<u>DEF 16</u> $S \subseteq X \times Y$ is a <u>fooling set</u> for
$f: \mathcal{X} \times Y \to \{0,1\}$ if $\exists\, z \in \{0,1\}$ s.t.

(a) $\forall\ (x,y) \in S \quad f(x,y) = z$

(b) $\forall\ (x_1, y_1) \neq (x_2, y_2) \in S$
either $f(x_1, y_2) \neq z$ or $f(x_2, y_1) \neq z$.

<u>LEM 17</u> If f has fooling set S ~~of~~ then
$D(f) \geq \log_2 |S|$

<u>Proof</u> Any P partitions $X \times Y \supseteq S$ into monochrom
rectangles. Claim: No rectangle contains more than
one $(x,y) \in S$. Hence #rect $\geq |S|$
Apply Cor 15.

Suppose rectangle R contains $(x_1,y_1), (x_2,y_2) \in S$.
Then by Prop 12 R contains (x_1, y_2) and
(x_2, y_1). But by def of fooling set
f does not have same value on these
4 inputs. $\boxed{\text{Q}}$

Optimize slightly:

Do argument in Lem 17 twice, once for 0-rectangles and once for 1-rectangles.

Ex 18 $\quad x, y \in \{0,1\}^n \quad EQ(x,y) = [x=y]$

$$S = \{ (\alpha,\alpha) \mid \alpha \in \{0,1\}^n \}$$

is fooling set of size 2^n for $\chi = 1$

$EQ(\alpha,\alpha) = 1 \quad EQ(\alpha,\beta) = 0$ for $\alpha \neq \beta$.

$\{ (0^n, 1^n), (1^n, 0^n) \}$ is fooling set of size 2 for $\chi = 0$

Lem 17 yields $D(EQ) > n$

Hence $D(EQ) = n+1$.

Ex 19 $\quad x, y \subseteq \{1, 2, \dots, n\}$

$DISJ(x,y) = [x \cap y = \emptyset]$

A fooling set of size 2^n is

$$S = \{ (A, \overline{A}) \mid A \subseteq \{1, 2, \dots, n\} \}$$

For $A \neq B$, either $A \cap \overline{B}$ or $\overline{A} \cap B$ is non-empty.

Hence $D(DISJ) \geq n$

Count also 0-rectangles $\Rightarrow D(DISJ) > n$

Trivial upper bound $D(DISJ) \leq n+1$.

Fooling set technique special case of following.

<u>LEM 20</u> Let μ probability distribution on $X \times Y$. If any monochromatic triangle has measure $\mu(R) \le \delta$, then $D(f) \ge \log_2(1/\delta)$

<u>Proof</u> $\mu(X \times Y) = 1$ by def. Hence at least $1/\delta$ rectangles in any monochrom partition.

For fooling set S, set $\mu((x,y)) = 1/|S|$ for $(x,y) \in S$, $\mu((x,y)) = 0$ if $(x,y) \notin S$.

Another application of Lem 20.

If all monochrom rectangles are small, say of size $\le k$, then # rect $\ge$ $|X||Y|/k$ and hence
$$D(f) \ge \log|X| + \log|Y| - \log k.$$

[Melhorn & Schmidt '82]

RANK LOWER BOUND

Given $f : X \times Y \to \{0,1\}$, associated matrix M_f of dim $|X| \times |Y|$ with (x,y)-entry $f(x,y)$.

Set of vectors $\vec{v}_1, \ldots, \vec{v}_m$ ~~in~~ $\mathbb{R}^n$ are linearly independent over $\mathbb{R}$ if ~~∀~~ for $a_i \in \mathbb{R}$

$$a_1 \cdot \vec{v}_1 + a_2 \cdot \vec{v}_2 + \ldots + a_m \vec{v}_m = \vec{0}$$

implies $a_1 = a_2 = \ldots = a_m = 0$.

Rank of M (over $\mathbb{R}$) = max # lin indep rows
= max # lin indep columns

DEF 21 $\text{rank}(f)$ = rank of M_f (over reals $\mathbb{R}$)

LEM 22 For any $f : X \times Y \to \{0,1\}$ $D(f) \geq \log_2 \text{rank}(f)$.

Proof Let $\mathcal{P}$ protocol, Let $\mathcal{L}_1$ 1-labelled leaves.

For every $\ell \in \mathcal{L}_1$ define M_ℓ by

$$M_\ell(x,y) = \begin{cases} 1 & \text{if } (x,y) \in R_\ell \\ 0 & \text{otherwise.} \end{cases}$$

Then $M_f = \sum_{\ell \in \mathcal{L}_1} M_\ell$

By linear algebra, $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$

Clearly $\text{rank}(M_\ell) = 1$

So $\text{rank}(f) = \text{rank}(M_f) \leq |\mathcal{L}_1| \leq |\mathcal{L}|$.

Now use Cor 15.

Sharpen this a bit

COR 23 $\quad D(f) \geq \log_2 (2 \cdot \mathrm{rank}(f) - 1)$

Proof Do Lem 22 for 1-rectangles.
then look at $\mathrm{not}(f) = 1 - f$

$\#$ 0-rectangles $\geq \mathrm{rank}(M_{\mathrm{not}(f)})$

But $M_{\mathrm{not}(f)} = J - M_f$

where J denotes all-ones matrix

Hence $\mathrm{rank}(M_f)$ & $\mathrm{rank}(M_{\mathrm{not}(f)})$
differ by at most 1 $\boxed{\text{q}}$

Ex 24 Look at function EQ again

M_{EQ} is identity matrix

$$\begin{pmatrix} 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & 0 \\ 0 & & & 0 & 1 \end{pmatrix}$$ with 2^n rows and columns

Rank $= 2^n \Rightarrow$ CC lower bound $n+1$.