## The General Setup

**Definition** A combining operator $\psi$ takes as input functions $f, g : \{0,1\}^n \longrightarrow Z$ and returns a function $h : \{0,1\}^n \longrightarrow R$. $\psi$ is simple if $\forall f, g$ and all $x \in \{0,1\}^n$, $h(x)$ can be computed given only $x$, $f(x)$, $g(x)$.

**Property Testing Communication Game**

Given property $P$, combining operator $\psi$, the communication game $C^P_\psi$ is the following:

Alice gets $f$
Bob gets $g$
Goal: Compute $C^P_\psi(f,g) := \begin{cases} 1 & \text{if } \psi(f,g) \in P \\ 0 & \text{if } \psi(f,g) \text{ is } \varepsilon\text{-far from } P. \\ * & \text{otherwise} \end{cases}$

$*$ means you can output 0 or 1.

**Main Reduction Lemma**

Fix finite set $Z$. For any property $P$ and any simple combining operator $\psi$

(1) $R(C^P_\psi) \le 2\,Q(P) \cdot \lceil \log |Z| \rceil$

(2) $R^0(C^P_\psi) \le 2\,Q'(P) \cdot \lceil \log |Z| \rceil$

(3) $R^{\rightarrow}(C^P_\psi) \le Q^{NA}(P) \cdot \lceil \log |Z| \rceil$

$R^0(f)$ complexity of computing $f$ with one-sided error where must be correct when $f(x,y)=1$

$R^{\rightarrow}(f)$ complexity of computing $f$ with one-way protocol
i.e. Alice sends single msg to Bob.

# Testing Monotonicity

Let $x, y \in \{0,1\}^n$

**Defn** $x \leq y$ if $x_i \leq y_i$ for all $i$.

i.e. if $x_1 \leq y_1$, $x_2 \leq y_2$, $\vdots$, $x_n \leq y_n$ then $x \leq y$

This is a partial ordering on $\{0,1\}^n$.

**Defn** Fix a finite set $R$.

A function $f: \{0,1\}^n \longrightarrow R$ is monotone if $x \leq y \Rightarrow f(x) \leq f(y)$.

$MONO := \{\text{monotone } f\}$

Query Complexity of MONO:

upper bounds: $O\left(\frac{n}{\varepsilon} \log|R|\right)$ [DGLRRS+ '99]

$O\left(\frac{n}{\varepsilon}\right)$ [Chakrabarty-Seshadhri 12]

lower bounds:

nonadaptive: $\Omega(\log n)$ [Fisher+ 02]

~~adap~~ nonadaptive, one-sided error: $\Omega(\sqrt{n})$ [Briet+ 10]

Adaptive: $\Omega(\log \log n)$, $\Omega(\log n)$ (one-sided error)

Today: $\Omega\left(\min\{n, |R|^2\}\right)$ [BBM 11]

Note: when $|R|$ large, this is $\Omega(n)$ vs $\Omega(\log \log n)$
old bound

State of the art: $\Omega\left(\min\left\{\frac{n}{\varepsilon}, \frac{|R|^2}{\varepsilon}\right\}\right)$ [Brody 12]

## The lower bound

(1) $R = \mathbb{Z}$. Reduce from DISJ

combining operator $\psi$: takes $f, g : \{0,1\}^n \to \{-1,+1\}$
returns $h : \{0,1\}^n \to \mathbb{Z}$
defined by $\boxed{h(x) = 2|x| + f(x) + g(x)}$

Lower bound for $Q(\text{MONO})$ follows from:

$$2Q(\text{MONO}) \geq R(C_{\psi}^{\text{MONO}}) \geq R(\text{DISJ}) = \Omega(n)$$

Main Reduction Lemma — To Do. — already known

---

$\text{DISJ} \longrightarrow C_{\psi}^{\text{MONO}}$ reduction:
Alice gets $A \subseteq [n]$. builds $f_A(x) := (-1)^{\sum_{i \in A} x_i}$
Bob gets $B \subseteq [n]$. builds $g_B(x) := (-1)^{\sum_{i \in B} x_i}$
Now let $h \longleftarrow \psi(f_A, g_B)$
Then $\boxed{h(x) := 2|x| + f_A(x) + g_B(x)}$

<u>Claim:</u> ① if $A \cap B = \emptyset$ then $h$ monotone
② if $A \cap B \neq \emptyset$ then $h$ $\frac{1}{8}$-far from monotone.

<u>proof</u> For $x \in \{0,1\}^n$, $i \in [n]$, $b \in \{0,1\}$ define $x^b \in \{0,1\}^n$ by: $x_j^b = \begin{cases} b & \text{if } j = i \\ x_j & \text{otherwise} \end{cases}$

The pair $(x^0, x^1)$ is called an edge of $\{0,1\}^n$ in the $i$th direction

An edge is <u>violating</u> if $\boxed{h(x^0) > h(x^1)}$

when can an edge violate monotonicity? must have $h(x^1) - h(x^0) < 0$.

$$h(x^1) - h(x^0) = 2|x^1| + f_A(x^1) + g_B(x^1) - 2|x^0| - f_A(x^0) - g_B(x^0)$$
$$= 2 + \left(f_A(x^1) - f_A(x^0)\right) + \left(g_B(x^1) - g_B(x^0)\right)$$

Note: if $i \notin A$ then

$$f_A(x') = (-1)^{\sum_{j\in A} x'_j} = (-1)^{\sum_{j\in A} x_j} = f(x^0)$$

(similar for $g_B$)

For $h(x') - h(x^0) < 0$ need $f_A(x') - f_A(x^0) = -2$

and $g_B(x') - g_B(x^0) = -2$

$\Rightarrow$ need $i \in A$ and $i \in B$ $\Rightarrow A \cap B \neq \emptyset$.

We saw that $A \cap B = \emptyset \to$ no edge violates monotonicity.

if $i \in A \cap B$ then $(x^0, x')$ violated whenever

$$f_A(x^0) = g_B(x^0) = 1$$
$$f_A(x') = g_B(x') = -1$$

this happens with probability $\frac{1}{2}$ for A and $\frac{1}{2}$ for B

in total both happen w/ prob $\frac{1}{4}$ total (or $\frac{1}{2}$ if $A = B$)

For each of these $\frac{2^n}{4}$ disjoint violating edges, need to change $h(x^0)$ or $h(x')$. $\Rightarrow$ Need to change at least $\frac{1}{8} \cdot 2^n$ entries of truth table to get monotone function

(2) $|R| = \Omega(\sqrt{n})$. Specifically say $|R| \geq 12\sqrt{n} + 5$.

<u>Lemma</u> There exists $\psi'$ such that for all $A, B \subseteq [n]$
if $h' \leftarrow \psi'(f_A, g_B)$ then ① $h'$ is monotone if $A \cap B = \emptyset$
② $h'$ is $\frac{1}{16}$-far from monotone if $A \cap B \neq \emptyset$

Testing Bipartiteness possible w/ $\tilde{O}(1/\varepsilon^3)$ queries

Algorithm

① Pick random $S \subseteq V$ $|S| = \tilde{O}(1/\varepsilon^2)$

② query $(u,v)$ for all $u,v \in S$

③ ACCEPT if there is valid partition
REJECT if $(S_1,S_2)$ not bipartite $\forall$ partitions $(S_1,S_2)$

Proof We'll actually analyze this alg:

① Pick ~~[illegible]~~ $\tilde{O}(1/\varepsilon)$ ~~vertices~~ $U \subseteq V$

② Pick $S = \tilde{O}(|U|/\varepsilon)$ edges
query. How to partition $S = S_1 \cup S_2$?

Before assume $G$ is $\varepsilon$-far from bipartite

Defn $v$ is influential if it has $\frac{\varepsilon N}{4}$ neighbors

Claim w/prob $\geq 5/6$, ~~at least~~ all but at most $\frac{\varepsilon N}{4}$ influential ~~[illegible]~~ vertices adjacent to $U$

proof Let $v$ be influential

$$\Pr[v \text{ not adj to } U] \leq (1-\varepsilon/4)^{|U|}$$

$$\leq e^{-\frac{|U|\varepsilon}{4}}$$

$$= e^{-\ln 24/\varepsilon}$$

$$= \frac{\varepsilon}{24}$$

$\boxed{\text{set } |U| = \frac{4\ln 24/\varepsilon}{\varepsilon}}$

$$E[\#v \text{ not adj to } U] \leq \frac{N\varepsilon}{24}$$

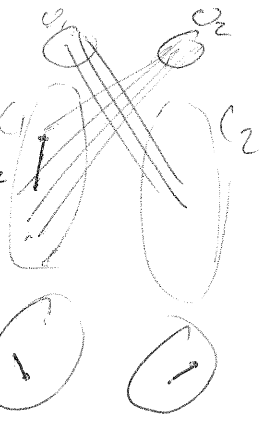
$$\Pr[\geq \tfrac{N\varepsilon}{4} \; v \text{ not adj to } U] \leq \frac{1}{6} \quad \text{(Markov)}$$

Now, Fix partition $U_1, U_2$ of $U$.

$C_1$ : neighbors of $U_2$

$C_2$ : neighbors of $U_1$

$R_1, R_2$ : arb. partition of remaining vertices




Claim: $\leq \frac{\varepsilon N^2}{2}$ violating edges touch $R = R_1 \cup R_2$

proof $\leq \frac{\varepsilon N}{4}$ influential $\times N$

$+ \leq N$ non-influential $\times \frac{\varepsilon N}{4}$

$= \frac{\varepsilon N^2}{2}$ total edges

Claim: For any fixed partition $U_1 \cup U_2$, $\Pr[\exists \text{ partition } S_1,S_2 \text{ that fails to witness nonbipartiteness}] \leq \frac{2^{-|U|}}{6}$

proof ~~[illegible]~~ There are $\geq \frac{\varepsilon N^2}{2}$ violating edges in $C$. $\boxed{|S| = \frac{16|U|}{\varepsilon}}$

We query $\geq 1$ w/ prob $\geq 1-(1-\frac{\varepsilon}{2})^{|S|/2} \geq 1-\frac{2^{-|U|}}{6}$

if we hit one violating pair $(v,w)$ how to partition it? w/og say $v,w \in C_2$

can't put ~~both~~ $v,w \in S_1$ or $v,w \in S_2$

~~if one is put in $S_1$ then it violates~~

If we put $v \in S_1$ then $(u,v)$ witnesses where $u \in U$ is $v$'s neighbor in $U_1$.

Corollary Prob that we find any partition that doesn't witness non-bipartiteness $\leq 2^{|U|} \cdot \frac{2^{-|U|}}{6}$

$= 1/6$

Overall success: $2/3$

(3) $|R| = o(\sqrt{n})$. Goal: show $\Omega(|R|^2)$ lower bound

Technique: dimension expansion.

Let $A$ be best algorithm for testing $f:\{0,1\}^n \to R$ for MONO

Let $m$ be greatest integer such that $\boxed{|R| \geq 12\sqrt{m} + 5}$

Create MONO testing alg $A'$ for functions $g:\{0,1\}^m \to R$ using $A$.

Note: $|R| = \Omega(\sqrt{m})$ so by (2) $A'$ requires $\Omega(m) = \Omega(|R|^2)$ queries.

<u>Claim</u> Given $g:\{0,1\}^m \to R$ there exists $f:\{0,1\}^n \to R$ such that
(i) if $g$ is monotone then $f$ is monotone
(ii) if $g$ is $\varepsilon$-far from monotone then $f$ is $\varepsilon$-far from monotone
(iii) $f(x)$ can be determined from one query to $g(x)$.

First let's use this claim to construct $A'$.
Let $f:\{0,1\}^n \to R$ be guaranteed by Claim.

$A'$ runs $A$ on $f$ and accepts iff $A$ accepts $f$.

Note: $g$ monotone $\Rightarrow f$ monotone $\Rightarrow A$ accepts w.h.p.
$g$ $\varepsilon$-far $\Rightarrow F$ $\varepsilon$-far $\Rightarrow A$ rejects w.h.p.

Also, $A'$ makes one query to $g$ for each query $A$ makes.

$$\begin{aligned}
\Rightarrow \quad Q(\text{MONO}(f)) &= \#\text{queries for } A \\
&= \#\text{queries for } A' \\
&\geq Q(\text{MONO}(g)) \\
&= \Omega(m) = \Omega(|R|^2).
\end{aligned}$$

Finally, let's prove Claim.

Fix $g : \{0,1\}^m \to R$

Define $f : \{0,1\}^n \to R$ by padding

write $f$ as $f : \{0,1\}^m \times \{0,1\}^{n-m} \to R$

$$f(x,y) := g(x).$$

Easy to see that (i) $g$ monotone $\Rightarrow$ $f$ monotone

(iii) $f$ computed from one query to $g$

(ii) not so simple.

Prove contrapositive: assume $f$ is <u>not</u> $\varepsilon$-far from monotone.

Let $\hat{f}$ be monotone function closest to $f$

$$\Rightarrow \Pr_{x,y}\left[\hat{f}(x,y) \neq f(x,y)\right] < \varepsilon$$

Now, let $\hat{y} := \min_{y} \Pr_{x}\left[\hat{f}(x,y) \neq f(x,y)\right]$

By averaging argument $\Pr_x\left[\hat{f}(x,\hat{y}) \neq f(x,\hat{y})\right] < \varepsilon.$

Define $\hat{g} : \{0,1\}^m \to R$ by

$$\hat{g}(x) := \hat{f}(x,\hat{y}).$$

Then ① $\hat{g}$ is monotone

② $\Pr_x\left[\hat{g}(x) \neq g(x)\right]$

$= \Pr_x\left[\hat{f}(x,\hat{y}) \neq f(x,\hat{y})\right]$

$< \varepsilon$

so $g$ is <u>not</u> $\varepsilon$-far from monotone.