G n-vertex d-regular graph is:

① EDGE EXPANDER if for every $S \subseteq V(G)$ $|S| \leq n/2$ $|E(S, \bar{S})| = \Omega(|S|) \geq \kappa|S|$ for some $\kappa > 0$

constant fraction of edges exit S

② VERTEX EXPANDER if for every $S \subseteq V(G)$, $|S| \leq \delta n$ for some constant $\delta > 0$,

$| N(S) | = | \{ u \mid \exists (u,v) \in E \text{ for } v \in S \} |$

$\geq \kappa|S|$ for some $\kappa > 1$

all moderately large vertex sets have large neighbourhood

③ SPECTRAL EXPANDER if $\tilde{A}_G$ has spectral gap bounded away from 0, i.e.

$1 - \lambda(G) > 0$

or $\geq \varepsilon$ for some constant $\varepsilon > 0$

These concepts are connected

Edge expansion $\Rightarrow$ large neighbour sets

Large neighbour set $\Rightarrow$ many outgoing edges

Already saw that large spectral gaps

$\Rightarrow$ quickly mixing random walks

$\Rightarrow$ graph has got to be well connected.

Recall $\tilde{A}_G$ has eigenvalues $\mu_1 = 1 \geq \mu_2 \geq \ldots \geq \mu_n$

$$\lambda(G) = \max_{2 \leq i \leq n} (|\mu_i|)$$

ASIDE
Bipartite graphs:

$\mu_n = -1$ spectral gap 0

But for connected bipartite graphs the "correct" definition of $\lambda(G)$ is $\lambda(G) = \max_{2 \le i \le n-1} (|\mu_i|)$

and if $1 - \lambda(G)$ is large, then G excellent bipartite expander.

However, for now focus on non-bipartite graphs

Actually, even here the literature seems slightly schizophrenic

And focus on connection between
- edge expansion
- spectral expansion

DEF 1 G is an (n, d, λ)-SPECTRAL EXPANDER GRAPH if

- $|V(G)| = n$
- G is d-regular
- $\lambda(G) \le \lambda$

$\{G_n\}_{n \in \mathbb{N}^+}$ is ~~an~~ spectral ~~expander~~ graph family if $\exists$ constants $d \in \mathbb{N}$, $\lambda < 1$ s.t. every G_n is an (n, d, λ)-spectral expander.

How large can the spectral gap be?

Always

$$\lambda \geq (1 - o_n(1)) \frac{2\sqrt{d-1}}{d}$$

Alon-Boppana bound.

Graphs meeting this bound are called RAMANUJAN GRAPHS.

We can easily prove a weaker bound.

PROPOSITION E 1/2

For an n-vertex d-regular graph G it holds that

$$\lambda \geq (1 - o_n(1)) \frac{\sqrt{d}}{d}$$

Recall that

$$\mathrm{trace}(A) = \sum_{i=1}^{n} a_{ii}$$

Standard fact

$$\mathrm{trace}(A) = \sum_{i=1}^{n} \mu_i$$

Note that $(\tilde{A}_G)^k_{ii}$ = (# walks of length k starting and ending in vertex i) / d^k

In particular $(\tilde{A}_G)^2_{ii} \geq d / d^2 = 1/d \quad \forall i$

take one of d edges and then go back

$$\mathrm{trace}(\tilde{A}_G^k) = \frac{1}{d^k} \left(\begin{array}{c} \text{\# walks of length } k \text{ starting} \\ \text{and ending at same vertex} \end{array} \right)$$

Hence

$$\operatorname{trace}(\tilde{A}_G^2) \geq n/d$$

On the other hand, since $\tilde{A}_G^2$ has eigenvalues μ_i^2, $i=1,\dots,n$ we get

$$\operatorname{trace}(\tilde{A}_G^2) = \sum_{i=1}^n \mu_i^2$$

$$= \sum_{i=1}^n \lambda_i^2$$

absolute values of eigenvalues sorted in order

$$\leq 1 + (n-1)\lambda^2$$

This yields

$$\lambda^2 \geq \frac{1}{d} \quad \frac{n-d}{n-1}$$

or $$\lambda \geq \frac{1}{\sqrt{d}}(1 - o_n(1)) = \frac{\sqrt{d}}{d}(1 - o_n(1))$$

as claimed $\boxed{}$

How common are expander graphs?
Very common - a random graph is an excellent expander almost surely

How get we get our hands on such graphs?
Often for applications explicit constructions are needed. This is more complicated.

A family of graphs $\{G_n\}_{n \in \mathbb{N}}$ is

a) EXPLICIT if there is a poly-time algorithm that on input $1^n = \underbrace{11\ldots1}_{n \text{ ones}}$ outputs the adjacency matrix of G (or random walk matrix of G

b) STRONGLY EXPLICIT if ∃ poly-time algo that on input (n, v, i) ($v \in [n]$, $i \in [d]$ degree) outputs index of ith neighbour of v.

[Note input of size $O(\log n)$ here, so algorith runs in time polylog(n).]

Several explicit constructions of expander graphs known

Some of these have

- very simple graph construction
- very deep and complicated analysis

<u>Example 2</u> Define for prime p and $x \in \mathbb{F}_p$

$$x^* = \begin{cases} 0 & \text{if } x = 0 \\ x^{-1} & \text{otherwise} \end{cases}$$

Let $G_p = (V_p, E_p)$ be defined by 3-regular graph

$V_p = \mathbb{F}_p$

Each $x \in \mathbb{F}_p$ has edges $(x, x+1)$, $(x, x-1)$, (x, x^*)

Analysis depends on deep result in number theory. See "Expander graphs and their applications" by Hoory, Linial, and Wigderson, Bulletin of the AMS 2006, Section 11.1.2 for details

Only (mildly) explicit. Limitation: Generation of primes.

<u>Example 3</u> $m \in \mathbb{N}^+$ G_m defined by

$V_m = \mathbb{Z}_m \times \mathbb{Z}_m$ ← integers mod m

(x,y) has edges to $(x+y, y)$, $(x-y, y)$, $(x, y+x)$, $(x, y-x)$, $(x+y+1, y)$, $(x-y+1, y)$, $(x, y+x+1)$, $(x, y-x+1)$

8-regular graph

all operations mod m

First explicit expander construction.

~~Strongly~~ explicit.

We will se a bit more complicated construction but with elementary analysis

Let us next define expansion in terms of edges and discuss connections to eigenvalues

DEF 4 The EDGE BOUNDARY ∂S of a vertex set $S \subseteq V(G)$ is $\partial S = E(S, \bar{S})$

DEF 5 G is an (n, d, ρ)-edge expander graph if

- $|V(G)| = n$
- G is d-regular
- $\min_{0 < |S| \leq n/2} |\partial S| / |S| \geq \rho$

DEF 6 The number

$$h(G) = \min_{0<|S|\leq n/2} \frac{|E(S, \bar{S})|}{|S|}$$

is know as the EGDE EXPANSION, or ISOPERIMETRIC NUMBER, or **CHEEGER CONSTANT** of G

$\{G_n\}_{n \in \mathbb{N}^+}$ is an edge expander graph family if $\exists$ constants $d \in \mathbb{N}$, $\rho > 0$ s.t. every G_n is an (n, d, ρ)-edge expander

[Note that Arora-Barak define edge expander with $g' = g/d$. Gives slightly nicer numbers, perhaps, but Defs 5 and 6 are more standard.]

Today we will prove:

THEOREM 7 (CHEEGER INEQUALITIES)
Let G be a d-regular n-vertex graph with eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ ~~Then~~ of $\bar{A}_G$. Then

$$\frac{d(1-\mu_2)}{2} \leq h(G) \leq \sqrt{2d^2(1-\mu_2)}$$

Cheeger proved continuous version of this. Discrete analogues due to [Dodziuk '84] and [Alon & Milman '85], [Alon '86]

Before doing so, let's try to gather some intuition

Edge expanders exist. (As we already said) Random graphs are likely to be excellent expanders, and can be used to prove the following

THEOREM 8 For any $\varepsilon > 0$ there exist $d, N \in \mathbb{N}^+$ such that for every $n > N$ there exists an $(n, d, d(1/2 - \varepsilon))$-edge expander.

Expansion $d/2$ ~~is~~ can be shown to be an upper bound.
If the spectral gap is large for G, then G shares some properties of random graphs as follows.

LEMMA 8 (EXPANDER MIXING LEMMA)

Let G be a d-regular n-vertex graph. Then for all $S, T \subseteq V$ it holds that

$$\left| \, |E(S,T)| - \frac{d}{n} |S||T| \, \right| \leq \lambda d \sqrt{|S||T|}$$

(somewhat) close if $\lambda(G)$ small

Edges between S and T

Expected # edges between S and T in random graph of edge density d/n

Proof Let 1_S and 1_T be characteristic vectors of S & T. Let $u_1, \ldots, u_n$ be orthonormal eigenbasis (where $u_1 = 1/\sqrt{n}$).

Write

$$1_S = \sum_i \alpha_i u_i$$

$$1_T = \sum_j \beta_j u_j$$

Then

$\frac{1}{d}$ for edge (i,j)

$$\frac{|E(S,T)|}{d} = \cancel{\frac{d}{d}} \sum_{i,j \in [n]} (1_S)_i \, a_{ij} \, (1_T)_j$$

$$= (1_S)^T \tilde{A}_G \, 1_T$$

$$= \left(\sum_i \alpha_i u_i\right)^T \tilde{A}_G \left(\sum_j \beta_j u_j\right)$$

$$= \sum_{i=1}^{n} \mu_i \alpha_i \beta_i$$

For a vector v, the coefficient α_i in

$$v = \sum_i \alpha_i u_i \quad \text{is} \quad \alpha_i = \langle v, u_i \rangle .$$

Hence

$$\alpha_1 = \langle 1_S , 1/\sqrt{n} \rangle = |S| / \sqrt{n}$$

$$\beta_1 = \langle 1_T , 1/\sqrt{n} \rangle = |T| / \sqrt{n}$$

and

$$\mu_1 \alpha_1 \beta_1 = \frac{|S|\,|T|}{n}$$

Then

$$\left| |\mathcal{E}(S,T)| - d \frac{|S||T|}{n} \right| =$$

$$d \left| \sum_{i=2}^{n} \mu_i \alpha_i \beta_i \right| \leq$$

$$d \sum_i | \mu_i \alpha_i \beta_i | \leq$$

$$d \lambda \sum_i | \alpha_i \beta_i | \leq \quad \text{[Cauchy-Schwarz]}$$

$$d \lambda \, \| \vec{\alpha} \|_2 \, \| \vec{\beta} \|_2 = \quad \text{[Orthonormality]}$$

$$d \lambda \, \| 1_S \|_2 \, \| 1_T \|_2 =$$

$$d \lambda \sqrt{|S|\,|T|}$$

$$\| 1_S \|_2^2 = \| \textstyle\sum_i \alpha_i u_i \|^2 = \sum_i \| \alpha_i u_i \|^2 = \sum_i \alpha_i^2$$

$$\vec{\alpha} = (\alpha_1, \dots, \alpha_n)^T$$
$$\vec{\beta} = (\beta_1, \dots, \beta_n)^T$$

$\boxed{\text{QED}}$

CAUCHY-SCHWARZ $\quad |\langle \vec{x}, \vec{y} \rangle| \leq \| \vec{x} \|_2 \, \| \vec{y} \|_2$

Proof of Cheeger inequalities follow

(1) Small 2nd eigenvalue $\Rightarrow$ edge expansion

Similar to Expander Mixing Lemma, but use that vertex sets are complementary. Prove (small expansion) $\Rightarrow$ large μ_2 $A = \tilde{A}_G$

First eigenvector is $\mathbb{1}$

Using Rayleigh quotients, we have that

$$\mu_2 = \max_{x \perp \mathbb{1}} \frac{x^T A x}{x^T x}$$

Want to find x yielding large RHS. How?

a) $x \perp \mathbb{1}$, so $\sum_{i=1}^{n} x_i = 0$

b) minimize $x^T x \Rightarrow$ spread out weights on all coordinates

c) Suppose S set with small cut $|E(S, \bar{S})|$
Natural to put positive weights on one set
negative weights on other

d) Don't know graph structure, so make all positive and negative coordinates equal

These considerations lead us to define

$$x = |\bar{S}|\, \mathbb{1}_S - |S|\, \mathbb{1}_{\bar{S}}, \text{ i.e. } \quad x_i = \begin{cases} |\bar{S}| & \text{if } i \in S \\ -|S| & \text{if } i \notin S \end{cases}$$

where we assume $|S| \le n/2$,

$$\frac{|E(S, \bar{S})|}{|S|} = h(G)$$

$x \perp \mathbb{1}$ since $\langle x, \mathbb{1} \rangle = \sum_i x_i = |S| \cdot |\bar{S}| + |\bar{S}|(-|S|) = 0$

$$x^T x = \|x\|_2^2 = |S|\,|\bar{S}|^2 + |\bar{S}|\,|S|^2 = |S||\bar{S}|(|S| + |\bar{S}|) = n|S||\bar{S}| \quad (1)$$

$\mathcal{E}\overline{X}$

$$x^T A x = \frac{2}{\cancel{\ }} \sum_{(i,j)\in \mathcal{E}} x_i a_{ij} x_j =$$

$(i,j) \in S^2$: Contribution $|\bar{S}|^2$

$(i,j) \in \bar{S}^2$: Contribution $|S|^2$

$(i,j) \in S \times \bar{S}$ Contribution $-|S||\bar{S}|$

Both positive

Negative but small contribution if cut is small

Denote $\mathcal{E}(U) = \{(v,w) \in \mathcal{E} \mid v,w \in U\}$

$$= \frac{2}{d}\left(|\mathcal{E}(S)|\,|\bar{S}|^2 + |\mathcal{E}(\bar{S})|\,|S|^2 - |\mathcal{E}(S,\bar{S})|\,|S||\bar{S}| \right)$$

Since G is d-regular we have

$$2|\mathcal{E}(S)| = d|S| - |\mathcal{E}(S,\bar{S})|$$
$$2|\mathcal{E}(\bar{S})| = d|\bar{S}| - |\mathcal{E}(S,\bar{S})|$$

$$= \frac{1}{d}\Big(|\bar{S}|^2 (d|S| - |\mathcal{E}(S,\bar{S})|) + |S|^2 (d|\bar{S}| - |\mathcal{E}(S,\bar{S})|) - 2|S||\bar{S}|\,|\mathcal{E}(S,\bar{S})| \Big)$$

Recall $n = |S| + |\bar{S}|$

$$= \frac{1}{d}\left(dn|S||\bar{S}| - \underbrace{n^2}_{=(|S|+|\bar{S}|)^2} |\mathcal{E}(S,\bar{S})| \right) \qquad (2)$$

This yields

$$\mu_2 \geq \frac{x^T A x}{x^T x}$$

$$\geq \frac{1}{d}\left(\frac{d n |S||\bar{S}| - n^2 |\mathcal{E}(S,\bar{S})|}{n|S||\bar{S}|}\right)$$

$$= 1 - \frac{1}{d} \cdot \frac{n |\mathcal{E}(S,\bar{S})|}{|S||\bar{S}|}$$ $\left[|\bar{S}| \geq n/2\right]$

$$\geq 1 - \frac{2}{d} \frac{|\mathcal{E}(S,\bar{S})|}{|S|}$$

$$= 1 - \frac{2}{d} h(G)$$

which is just a rearrangement of

$$\frac{d(1-\mu_2)}{2} \leq h(G)$$

This was the easy part...

(2) Edge expansion implies small 2nd eigenvalue

We want to prove $h(G) \le \sqrt{2d^2(1-\mu_2)}$

Switching to Arora-Barak notation/convention, this is the same thing as saying that if G is an $(n, d, \underbrace{\rho d}_{h(G)})$-edge expander, then

$$\boxed{1 - \rho^2/2 \ge \mu_2} \qquad (3)$$

Let u_2 be eigenvector associated to μ_2

Large 2nd eigenvalue should imply $\exists$ small cut.

Suppose u_2 contained only two values (if so: one positive, one negative, since $\sum_i (u_2)_i = 0$)

Then we could let $S = \{ i \mid (u_2)_i \ge 0 \}$

$\bar{S} = \{ i \mid (u_2)_i < 0 \}$

and argue as in (1).

Since Rayleigh quotient large, $|E(S, \bar{S})|$ must be small.

This is still a good idea in the general case, although the bound we get will get $\sqrt{1-\mu_2}$ instead of $(1-\mu_2)$ as a factor.

<u>Side note</u> We will see that the normalized Laplacian actually makes a cameo appearance in the proof — sort of strengthening the argument that this is a natural concept to study — but we will mostly ignore this...

Let u be such that

$$Au = \mu_2 u$$

positive part

negative part

Write $u = v + w$ where

$$v_i = \begin{cases} u_i & \text{if } u_i \geq 0 \\ 0 & \text{o/w} \end{cases} \qquad w_i = \begin{cases} 0 & \text{if } u_i \geq 0 \\ u_i & \text{o/w} \end{cases}$$

Note $v \neq \vec{0} \neq w$ since $u \perp 1$.

Suppose v is nonzero on <u>at most</u> $n/2$ coordinates (otherwise look at $-u$).

It is straightforward to verify that (3) follows from the following

$$\boxed{\begin{array}{l} \text{Let } \quad Z = \displaystyle\sum_{i,j \in [n]} a_{ij} \left| v_i^2 - v_j^2 \right| \\ \text{then:} \\ \underline{\text{Claim A}} \quad Z \geq 2g \, \|v\|_2^2 \\ \underline{\text{Claim B}} \quad Z \leq \sqrt{8(1-\mu_2)} \, \|v\|_2^2 \end{array}}$$

<u>Proof of Claim A</u>

Wlog, sort v so that $v_1 \geq v_2 \geq \dots \geq v_n$, with $v_i = 0$ for $i > n/2$

[BTW, why can we do this w.l.o.g.?]

Then for $i < j$ we have $|v_i^2 - v_j^2| =$

$$\cancel{|v_i^2 - v_j^2|} = v_i^2 - v_j^2 = \sum_{k=i}^{j-1} \left(v_k^2 - v_{k+1}^2 \right)$$

and using this we get

$$\mathcal{L} = \sum_{i,j\in[n]} a_{ij}\left(v_i^2 - v_j^2\right)$$

$$= 2\sum_{i<j} a_{ij} \sum_{k=i}^{j-1} v_k^2 - v_{k+1}^2 \qquad (4)$$

Each term $(v_k^2 - v_{k+1}^2)$ appears in (4) once per edge (i,j) s.t. $i \le k < j$ [i.e. when $a_{ij} \neq 0$] and the weight is then $2/d$. Furthermore $v_k = 0$ for $k > n/2$.

Changing the order of summation, we can therefore rewrite (4) as

$$\mathcal{L} = \frac{2}{d}\sum_{k=1}^{n/2} \left|E\left(\{1,\dots,k\},\{k+1,\dots,n\}\right)\right|\left(v_k^2 - v_{k+1}^2\right)$$

$$\geq \frac{2}{d}\sum_{k=1}^{n/2} \rho d k \left(v_k^2 - v_{k+1}^2\right) \qquad (5)$$

since G has edge expansion ρd.

Rearranging (5) and using $v_k = 0$ for $k > n/2$ again we get

$$(5) = 2\rho\left(\sum_{k=1}^{n/2} k v_k^2 - \sum_{k=1}^{n/2}(k-1)v_k^2\right)$$

$$= 2\rho \sum_{k=1}^{n} v_k^2$$

$$= 2\rho \, \|v\|_2^2$$

which establishes Claim A.

Proof of Claim B

Let us keep in mind that

$$Au = \mu_2 u \; ; \; u = v + w \; ; \; \langle v, w \rangle = 0$$

We claim that the Rayleigh quotient of v is

$$\frac{v^T A v}{v^T v} \geq \mu_2 \qquad (6)$$

To see this, write

$$\frac{v^T A v}{v^T v} = \frac{\langle Av, v \rangle}{\|v\|_2^2} \qquad [\text{A symmetric}]$$

$$\geq \frac{\langle Av, v \rangle + \langle Aw, v \rangle}{\|v\|_2^2} \qquad [\text{only negative terms}]$$

$$= \frac{\langle A(v+w), v \rangle}{\|v\|_2^2}$$

$$= \frac{\langle Au, v \rangle}{\|v\|_2^2}$$

$$= \frac{\langle \mu_2 (v+w), v \rangle}{\|v\|_2^2}$$

$$= \mu_2 \left(\frac{\|v\|_2^2 + \overbrace{\langle w, v \rangle}^{0}}{\|v\|_2^2} \right)$$

$$= \mu_2$$

It follows that [using $\tilde{\mathcal{L}}_G = I - \tilde{A}_G$

and $x^T \tilde{\mathcal{L}}_G x = \frac{1}{d} \sum_{(i,j) \in E} (x_i - x_j)^2$]

$$
\begin{aligned}
1 - \mu_2 &\geq 1 - \frac{v^T A v}{v^T v} \\
&= \frac{v^T I v}{v^T v} - \frac{v^T A v}{v^T v} \\
&= \frac{v^T \tilde{\mathcal{L}}_G v}{v^T v} \\
&= \frac{\sum_{i,j \in n} a_{ij} (v_i - v_j)^2}{2 \, \|v\|_2^2} \qquad (7)
\end{aligned}
$$

[This can also be verified directly by computing

$$\sum_{i,j} a_{ij} (v_i - v_j)^2 = \sum_{i,j} a_{ij} v_i^2 - 2 \sum_{i,j} a_{ij} v_i v_j + \sum_{i,j} a_{ij} v_j^2$$

$$= 2 \|v\|_2^2 - 2 \, v^T A v$$

using that rows and columns of A sum to one]

Multiply numerator and denominator in (7) by

$$\sum_{i,j \in [n]} a_{ij} \left(v_i + v_j \right)^2$$

E XVII

Let us index over $(i,j) \in [n]^2$
and define x & y by

$$x_{ij} = \sqrt{a_{ij}}\,|v_i - v_j| \qquad y_{ij} = \sqrt{a_{ij}}\,(v_i + v_j)$$

Then using Cauchy-Schwarz

$$|\langle x, y\rangle| \leq \|x\|_2 \|y\|_2$$ on the

new numerator yields

$$\left(\sum_{i,j} a_{ij}(v_i - v_j)^2\right)\left(\sum_{i,j} a_{ij}(v_i + v_j)^2\right) \geq$$

$$\geq \left(\sum_{i,j} a_{ij}\,|v_i - v_j|\,(v_i + v_j)\right)^2$$

$$= \left(\sum_{i,j} a_{ij}\,|v_i^2 - v_j^2|\right)^2 = Z^2 \qquad (8)$$

The denominator in (7) becomes

$$2\|v\|_2^2 \sum_{i,j} a_{ij}\,(v_i + v_j)^2 =$$

rows and columns sum to 1

$$= 2\|v\|_2^2\left(\sum_{i,j} a_{ij} v_i^2 + 2\sum_{i,j} a_{ij} v_i v_j + \sum_{i,j} a_{ij} v_j\right)$$

$$= 2\|v\|_2^2\left(2\|v\|_2^2 + 2\, v^T A v\right)$$

$$\leq 2\|v\|_2^2\left(4\|v\|_2^2\right)$$

since $v^T A v \leq \|v\|_2^2$
since max eigenvalue is 1

$$= 8\|v\|_2^4 \qquad (9)$$

Combining (8) and (9) with (7) we obtain

$$1-\mu_2 \geq \frac{Z^2}{8\,\|v\|_2^4}$$

and hence

$$Z \leq \sqrt{8(1-\mu_2)}\;\|v\|_2^2$$

and Claim B follows.

This establishes Thm 7 $\square$

① The ~~Lower~~ bound in the Cheeger inequalities is tight

Hypercube, d-dimensional: Vertices $\{0,1\}^d$

Edges between vertices that differ in one coordinate

$$h(G) = 1 \qquad 1-\mu_2 = 2/d$$

② The upper bound is also tight

n-vertex cycle: Vertices $\{0, 1, 2, \dots, n-1\}$

Edges from i to $i+1 \pmod n$ and $i-1 \pmod n$

$$h(G) = \Theta\left(\frac{1}{n}\right) \qquad 1-\mu_2 = \Theta\left(\frac{1}{dn^2}\right)$$