DD2442 AUTUMN 2014 LECTURE 7

Last lecture:

(1) Repeat probabilistic algorithm to drive down error probability exponentially

Use highly dependent random samples to get qualitatively similar error reduction while saving big on randomness

How? Take a walk on an expander

(2) Started building (strongly) explicit expander

Build recursively using three different graph products

(A) <u>Matrix product</u>

G (n, d, λ) - spectral expander
G' (n', d', λ') - spectral expander

$n = n'$

$G \cdot G'$ graph with normalized adjacency matrix $\tilde{A}_{G \cdot G'} = \tilde{A}_{G'} \cdot \tilde{A}_G$

\# vertices : n same
degree : dd' increases
expansion : improves

(B) <u>Tensor product</u>

$G \otimes G'$ graph with normalized adjacency matrix

$$\tilde{A}_{G \otimes G'} = \tilde{A}_G \otimes \tilde{A}_{G'} =$$

$$\begin{pmatrix} a_{11} \tilde{A}_{G'} & a_{12} \tilde{A}_{G'} & \cdots & a_{1n} \tilde{A}_{G'} \\ a_{21} \tilde{A}_{G'} & a_{22} \tilde{A}_{G'} & \cdots & a_{2n} \tilde{A}_{G'} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} \tilde{A}_{G'} & a_{n2} \tilde{A}_{G'} & \cdots & a_{nn} \tilde{A}_{G'} \end{pmatrix}$$

(Tensor product continued)

# vertices	nn'	increases
degree	dd'	increases
expansion	doesn't get worse	

ⓒ <u>Replacement product</u>

$d = n'$ (degree in G = # vertices in G')

d' constant (our design choice; definition works for any d')

*** To be described today ***

# vertices	$nn' = nd$	
degree	$2d'$	constant
expansion	doesn't get too much worse	

<u>Game plan</u>

Build sequence of expanders $(G(i))$ as follows

(i) Let $G(1)$ = excellent constant-size expander (if nothing else, can find using brute force)

for $i = 1, 2, 3, \ldots$

(ii) $G^{(1)} = G(i) \otimes G(i)$ [increase size]

(iii) $G^{(2)} = \underbrace{G^{(1)} \circ G^{(1)} \circ \cdots \circ G^{(1)}}_{\text{suitably many times}}$ [improve exp but increases degree]

another constant-size excellent expander

(iv) $G(i+1) = G^{(2)}$ Ⓡ H [decrease degree still good expansion]

Last time we proved

LEMMA 1

If G is an (n, d, λ)-spectral expander, then G^2 is an (n, d^2, λ^2)-spectral expander

LEMMA 2

If G (n, d, λ)-spectral expander and G' (n', d', λ')-spectral expander, then $G \otimes G'$ $(nn', dd', \max(\lambda, \lambda'))$-spectral expander

We also introduced graph representation ROTATION MAP $\widehat{G} : [n] \times [d] \to [n] \times [d]$

Number neighbours of each vertex $1, 2, \ldots, d$

$\widehat{G}(u, i) = (v, j)$ if

(a) v ith neighbour of u
(b) u jth neighbour of v

symmetry constraint $\Downarrow$ graph undirected

$\widehat{G}(\widehat{G}(u, i)) = (u, i)$

$\widehat{G}$ is a permutation on $[n] \times [d]$

We saw that $\widehat{G^2}$ and $\widehat{G \otimes G'}$ easy to compute from $\widehat{G}$, $\widehat{G'}$

Why this representation? Convenient to describe REPLACEMENT PRODUCT, which we will do next

Replacement product

G (n, D, λ)-graph
G' (D, d, λ')-graph

G Ⓡ G' has
vertices nD
degree $2d$
spectral expansion: to be determined

1) For every vertex u of G, make copy "cluster" of G' (including vertices and edges)

2) If $\widehat{G}(u,i) = (v,j)$, make d parallel edges between ith vertex in u-copy of G' and jth vertex in v-copy of G'

[Some versions ~~et~~ only have single edge in 2]

With d parallel edges, get that random step stays in same cluster with prob 1/2 moves to neighbouring cluster with prob 1/2

Rotation map for G Ⓡ G'

nD vertices: index by $(u,v) \in [n] \times [D]$
$2d$ edges: index by $(i, \sigma) \in [d] \times \{0,1\}$

$\sigma = 0$ stay inside cluster (treat v as vertex of G')

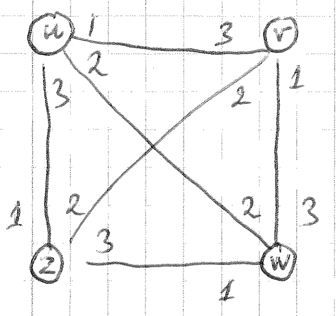
$$\widehat{G \circledR G'}((u,v),(i,0)) = (u, \widehat{G'}(v,i), 0)$$

$\sigma = 1$ move to neighbouring cluster (treat v as edge label of G)

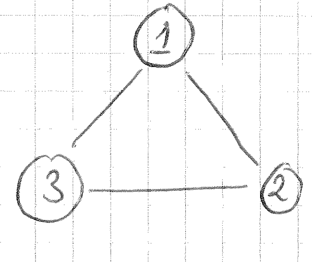
$$\widehat{G \circledR G'}(u,v,i,1) = (\widehat{G}(u,v), i, 1)$$

V

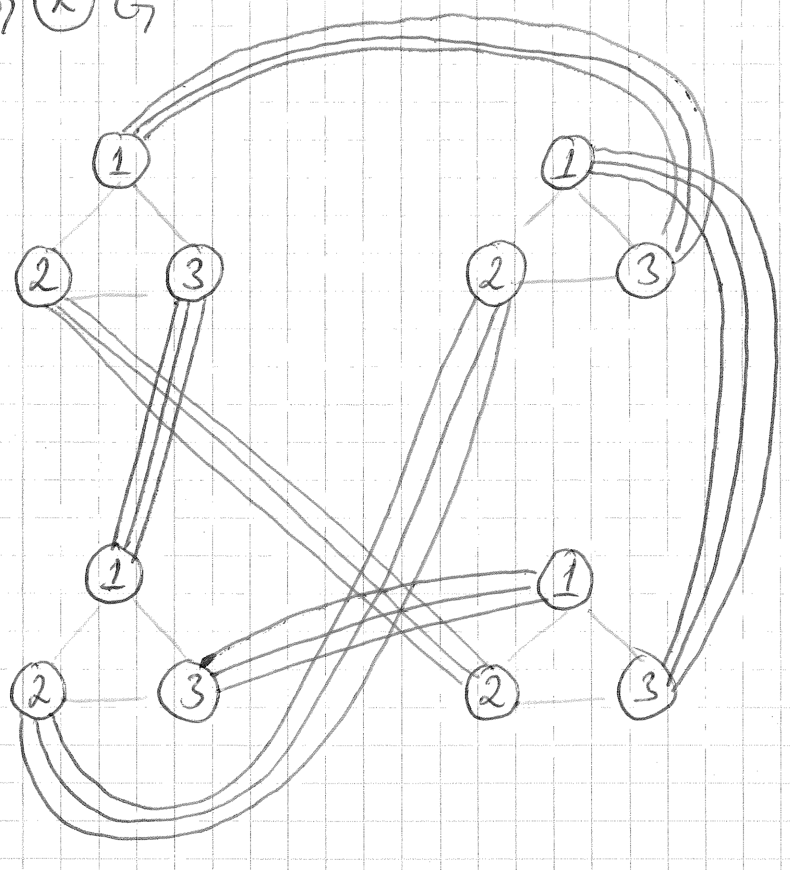
G



G'



$G \circledR G'$



$\widehat{G}(u,1) = (v,3)$

$\widehat{G}(u,2) = (w,2)$

$\widehat{G}(u,3) = (z,1)$

$\widehat{G}(v,1) = (w,3)$

$\widehat{G}(v,2) = (z,2)$

$\widehat{G}(v,3) = (u,1)$

$\widehat{G}(w,1) = (z,3)$

$\widehat{G}(w,2) = (u,2)$

$\widehat{G}(w,3) = (v,1)$

$\widehat{G}(z,1) = (u,3)$

$\widehat{G}(z,2) = (v,2)$

$\widehat{G}(z,3) = (w,1)$

Random-walk matrix / normalized adjacency matrix

Again index vertices $(u,i) \in [n]\times[D]$

0-edges inside same cluster described by

$$I_n \otimes \widetilde{A}_{G'} = \begin{pmatrix} \widetilde{A}_{G'} & 0 & \cdots & 0 \\ 0 & \widetilde{A}_{G'} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \widetilde{A}_{G'} \end{pmatrix}$$

1-edges described by $\widehat{G}$ written as permutation matrix

$$\widehat{A}\left((u,i),(v,j)\right) = \begin{cases} 1 & \text{if } \widehat{G}(u,i) = (v,j) \\ 0 & \text{otherwise} \end{cases}$$

(Why is this matrix even symmetric?
Because $\widehat{G}(u,i) = (v,j)$ iff $\widehat{G}(v,j) = (u,i)$
by def of rotation map)

Adding these matrices and scaling by $1/2$ to normalize, we get

$$\widetilde{A}_{G Ⓡ G'} = \frac{1}{2}\widehat{A} + \frac{1}{2}\left(I_n \otimes \widetilde{A}_{G'}\right) \qquad (\dagger)$$

We will do replacement product with fixed graph H having fixed degree d
$\Rightarrow$ bring degree down to constant again

What happens to expansion?

LEMMA 3

If G $(n, D, 1-\varepsilon)$ - spectral expander
and H $(D, d, 1-\delta)$ - spectral expander,
then

$$G \circledR H \quad \left(nD, 2d, 1 - \frac{\varepsilon\delta^2}{24}\right) \text{ - spectral expander.}$$

Intuition

G good expander, but degree D too high
k-step random walk on G requires
$O(k \log D)$ random bits

Want to use less randomness. How?
Randomness reduction using expander walk on G'
G' has D vertices — needed domain for edge labels
degree $d' \ll D$ - much less randomness

At each step of walk flip fair coin
- With 50% prob take random step on G'
- With 50% prob use random label from G' to pick edge in G

Will need some more technicalities to prove lem 3

DEF 4 The SPECTRAL NORM $\|A\|$ of an $n \times n$ matrix A
is $\max \|Av\|_2$ for $v \in \mathbb{R}^n$, $\|v\|_2 = 1$

For our random-walk matrices the norm is just the absolute value of the largest eigenvalue, i.e., 1

Straightforward to prove this is indeed a norm

In addition $\|AB\| \leq \|A\| \, \|B\|$

LEMMA 5

Let G (n, d, λ) - spectral expander with random-walk matrix $A = \tilde{A}_G$

Let J be the all-ones matrix and let $\frac{1}{n} J$ be the random-walk matrix of the n-clique with self-loops added. Then

$$A = (1-\lambda)\frac{1}{n}J + \lambda C$$

for some C such that $\|C\| \leq 1$

Proof

Define $C = \frac{1}{\lambda}\left(A - (1-\lambda)\frac{1}{n} J \right)$

Need to prove $\| C v \|_2 \leq \|v\|_2 \quad \forall v \in \mathbb{R}^n$

Decompose $v = u + w \qquad u = \alpha \mathbb{1}$ for some $\alpha \in \mathbb{R}$

$w \perp \mathbb{1}$

Since $A \mathbb{1} = \mathbb{1}$

and $\frac{1}{n} J \mathbb{1} = \mathbb{1}$

we get $Cu = \frac{1}{\lambda}\left(u - (1-\lambda) u \right) = u \qquad (1)$

Let $w' = Aw$

Then $\|w'\|_2 \leq \lambda \|w\|_2 \qquad (2)$

and since A maps $\mathbb{1}^\perp$ into itself we have

$w' \perp \mathbb{1}$

~~Thus~~ Also $Jw = 0$ since $\sum_i w_i = 0$

Hence

$$Cw = \frac{1}{\lambda}\left(A - (1-\lambda)\frac{1}{n} J\right) w$$

$$= \frac{1}{\lambda} w' \qquad (3)$$

Since $u \perp w'$ we get

$$\|Cv\|_2^2 = \|u + \frac{1}{\lambda} w'\|_2^2 \qquad [\text{by (1) \& (3)}]$$

$$= \|u\|_2^2 + \|\frac{1}{\lambda} w'\|_2^2 \qquad [\text{Pythagoras}]$$

$$\le \|u\|_2^2 + \|w\|_2^2 \qquad [\text{by (2)}]$$

$$\le \|v\|_2^2$$

and it follows that $\|C\| \le 1$ $\boxed{L}$

<u>Proof of Lem 3</u>

If we want to prove $K \le 1 - x$ for $|x| \le 1$, then sufficient to prove $K^3 \le 1 - 3x$ since $1 - 3x \le (1-x)^3$

So we show $\lambda(G Ⓡ H)^3 \le 1 - \frac{\varepsilon\delta^2}{8}$

By Lem 1 $\lambda(G Ⓡ H)^3 = \lambda\left((G Ⓡ H)^3\right)$

Let $\hat{A}$ ~~random-walk~~ permutation matrix ~~of~~ corresponding to $\hat{G}$

Let B random-walk matrix of H

Let A random-walk matrix of G

Let C random-walk matrix of $(G Ⓡ H)^3$

Then by (†) we have

$$C = \left(\frac{1}{2}\hat{A} + \frac{1}{2}(I_n \otimes B)\right)^3$$

If we expand this out, we get 8 terms like

$$\frac{1}{8}\underbrace{\hat{A}^3}$$

$$\frac{1}{8}\underbrace{\hat{A}^2 (I_n \otimes B)}$$

etc term. All of these are random matrices and thus have spectral norm ≤ 1

Summing up, we can write

$$C = \frac{7}{8} C' + \frac{1}{8}(I_n \otimes B)\hat{A}(I_n \otimes B) \qquad (*)$$

for $\|C'\| \leq 1$. Want to prove that second term shrinks vectors v, $v \perp 1$, by some nontrivial factor

Now use that by Lem 5

$$B = (1-\delta)B' + \frac{\delta}{D} J_D \qquad (**)$$

for $\|B'\| \leq 1$. Then plugging (**) into 2nd term of (*):

$$\frac{1}{8}(I_n \otimes B)\hat{A}(I_n \otimes B) =$$

$$= \frac{1}{8}\left(I_n \otimes \left((1-\delta)B' + \frac{\delta}{D}J_D\right)\right)\hat{A}\left(I_n \otimes \left((1-\delta)B' + \frac{\delta}{D}J_D\right)\right) \qquad (‡)$$

We need that if $\|B'\| \leq 1$ then $\|I_n \otimes B'\| \leq 1$. why? Gathering terms again, we get that $(\ddagger)$ can be written

$$\frac{1}{8}\frac{\delta^2}{\cancel{D^2}}\left(I_n \otimes \frac{J_D}{D}\right)\hat{A}\left(I_n \otimes \frac{J_D}{D}\right) + \frac{1}{8}(1-\delta^2)\, C'' \quad (\star)$$

for $\|C''\| \leq 1$

Combining $(*)$ and $(\star)$ we get

$$C = \left(1 - \frac{\delta^2}{8}\right) C''' + \frac{\delta^2}{8}\left(I_n \otimes \tfrac{1}{D} J_D\right)\hat{A}\left(I_n \otimes \tfrac{1}{D} J_D\right) \quad (\star\star)$$

for $\|C'''\| \leq 1$

We claim

<u>Claim A</u> $\#$ $\lambda\left(\left(I_n \otimes \frac{1}{D} J_D\right)\hat{A}\left(I_n \otimes \frac{1}{D} J_D\right)\right) \leq 1 - \varepsilon$

By Claim A and the triangle inequality, we get that for any v, $v \perp 1$, $\|Cv\|_2$ shrinks compared to $\|v\|_2$ by factor

$$\left(1 - \frac{\delta^2}{8}\right) 1 + \frac{\delta^2}{8}(1 - \varepsilon) = 1 - \frac{\delta^2 \varepsilon}{8}$$

which is exactly what we wanted to show.

Claim A in turn follows from

Claim B
$$\left(I_n \otimes \frac{1}{D} J_D\right) \hat{A} \left(I_n \otimes \frac{1}{D} J_D\right) = A \otimes J_D$$

Proof of Claim B

Both sides describe random walks on graphs on nD vertices. Want to prove that graphs are the same. Look first at LHS

Here we take one step from (u,i) by

1) Pick random $k \in [D]$ and move to (u,k), (because (u,i) has edges precisely to all other (u,j) including $j=i$)
2) Move from (u,k) to $\hat{G}(u,k) = (v,j)$
3) Pick random $j' \in [D]$ and move to (v,j')

On RHS the random walk takes one step from (u,i) by

1) Pick $j' \in [D]$ randomly.
2) Pick v as random neighbour in G and go to (v,j')

This is the same walk, so graphs are equal. Claim B follows $\square$

Perhaps another, more formal, way of looking at RHS in Claim B:

Let $e_u \in \mathbb{R}^n$ indicator vector for u
$e_i \in \mathbb{R}^D$ indicator vector for i

Want to see where we can go from (u, i)
This is described by

$$\left(A \otimes \frac{1}{D} J_D\right)\left(e_u \otimes e_i\right) =$$

$$(A e_u) \otimes \left(\frac{1}{D} J_D\, e_i\right)$$

Which is choosing a neighbour of u in G pluse a completely random $j \in [D]$.

$\boxed{\text{Claim A}}$ Proof of

We want to bound $\lambda\left(A \otimes \frac{1}{D} J_D\right)$
which is $\max\left(\lambda(A), \lambda\left(\frac{1}{D} J_D\right)\right)$

But $\lambda(J_D) = 0$

Suppose $v \perp 1$, i.e. $\sum_i v_i = 0$

Then $(J_D v)_i$ just sums up coordinates

and $(J_D v)_i = 0 \quad \forall i \in [D]$

Hence $\lambda(J_D) = \max\limits_{v \perp 1} \left|\frac{v^T J_D v}{v^T v}\right| = \left|\frac{v^T \vec{0}}{v^T v}\right| = 0$

and so $\lambda\left(A \otimes \frac{1}{D} J_D\right) = \lambda(A) \leq 1 - \varepsilon$ by assumption $\boxed{\text{Z}}$

Putting the pieces together

Now we can (semi-) describe a strongly explicit expander graph construction and prove its properties

THEOREM 6
There exist constants $d \in \mathbb{N}^+$, $\lambda < 1$ such that a strongly explicit family of degree-d graphs with spectral expansion λ exist

Remark 7
Once we have this, can get expansion for any λ (but then d depends on λ).
Just use matrix product

We will actually prove something slightly weaker, namely that there is a $c \in \mathbb{N}^+$ such that for every c^k, $k \in \mathbb{N}^+$ we can find a strongly explicit expander on c^k vertices. This can then be improved to yield Thm 6, but we might not have the energy to do that.

We prove Thm 6 by finding 3 base case expanders with good enough spectral expansion and then following our game plan.

Let $D = (2d)^{100}$ for d chosen large enough. Find by brute force

- H $(D, d, 0{,}01)$ - spectral expander
- G_1 $(D, 2d, 1/2)$ - spectral expander
- G_2 $(D^2, 2d, 1/2)$ - spectral expander

For $G_k > 2$ let

$$G_k = \left(G_{\lfloor \frac{k-1}{2} \rfloor} \otimes G_{\lceil \frac{k-1}{2} \rceil} \right)^{50} ⓡ H$$

LEMMA 8

For every $k \in \mathbb{N}^+$ G_k is a $((2d)^{100k}, 2d, 1 - \frac{1}{50})$ - spectral expander. Furthermore, there is a poly(k) - time algorithm that given k, i, j finds the jth neighbour of i in G_k.

Proof

The algorithm can be reconstructed from our discussions how to compute rotation maps for our different graph products plus using recursion. We skip the details.

Let us verify the parameters of G_k. Base cases G_1 and G_2 are clearly OK.

VERTICES

Let $n_k = |V(G_k)|$

$$n_k = n_{\lfloor (k-1)/2 \rfloor} \, n_{\lceil (k-1)/2 \rceil} \cdot \cancel{2d} (2d)^{100} = \left[\begin{array}{c} \text{induction} \\ \text{hypothesis} \end{array} \right]$$

$$= (2d)^{100 \lfloor (k-1)/2 \rfloor} \cdot (2d)^{100 \lceil (k-1)/2 \rceil} (2d)^{100}$$

$$\left\lceil \frac{k-1}{2} \right\rceil + \left\lfloor \frac{k-1}{2} \right\rfloor = k-1$$

$$= (2d)^{100(k-1)+100} = (2d)^{100k}$$

DEGREE

$G_{k'}$ $k' < k$ have degree $2d$

$G_{\lfloor \frac{k-1}{2} \rfloor} \otimes G_{\lceil \frac{k-1}{2} \rceil}$ has degree $(2d)^2$

Taking matrix product 50 times
yields degree $((2d)^2)^{50} = (2d)^{100}$

This is $= |V(H)|$

Taking replacement product brings degree
down to $2d$ again

$$\lambda(G_{k'}) \le 1 - \frac{1}{50} \quad \text{by induction hypothesis}$$

$$\lambda\left(G_{\lfloor \frac{k-1}{2} \rfloor} \otimes G_{\lceil \frac{k-1}{2} \rceil}\right) \le 1 - \frac{1}{50} \quad \text{by Lem 2}$$

$$\lambda\left(\left(G_{\lfloor \frac{k-1}{2} \rfloor} \otimes G_{\lceil \frac{k-1}{2} \rceil}\right)^{50}\right) \le \left(1 - \frac{1}{50}\right)^{50} \quad \text{by Lem 1}$$

Use

$$1 + x \le e^x$$

to derive

$$\left(1 - \frac{1}{50}\right)^{50} \le \frac{1}{e} < \frac{1}{2}$$

By Lem 3 we have

$$\lambda(G_k) \le \cancel{1 - (1/2)\, 1/100}$$

$$1 - \frac{(1/2)\,(99/100)^2}{24}$$

$$\le 1 - \frac{1}{50}$$

which proves Lem 8.

To get Thm 6, show that (n, d, λ)-expander can be transformed into (n', cd, λ')-expander for any $n/c \le n' \le n$, $\lambda' = \lambda'(\lambda, d)$ by joining together sets of at most c vertices into "mega-vertices".

Final remarks

Quantitative bounds in Theorem 6 pretty bad:
- Relation degree - expansion
- Running time

Partly because of simplified exposition
Further optimizations possible.
But even then this construction is not best known.

However:
- Analysis completely elementary
- This particular construction has found several applications elsewhere in TCS (which we won't have time/energy to study during this course)