Lecture 16: "PHP is hard for bd Frege" - part IV -

We conclude the proof of the following theorem.

Thm: Let $\mathcal{F}$ be a Frege system over $\{\vee, \neg\}$ and let $d > 3$. For sufficiently large n, every depth d proof of $\neg\, OFPHP_n^{n+1}$ in $\mathcal{F}$ has size $\geq 2^{n^\delta}$ for $0 < \delta < \left(\frac{1}{5}\right)^d$.

Lemma 3: Let d be an integer, $0 < \varepsilon < \frac{1}{5}$, $0 < \delta < \varepsilon^d$ and Γ a set of formulas of depth $\leq d$ closed under subformulas.
If $|\Gamma| < 2^{n^\delta}$ then there exist a $\rho \in M_n^q$ with $q = n^{\varepsilon^d}$ and there exist a $2n^\delta$-evaluation of $\Gamma|_\rho$.

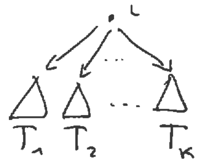
M_n^q = the set of all matchings of size q over P, H of size $n+1$ and n resp.

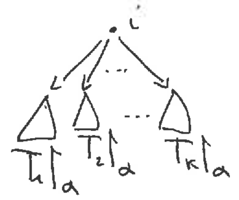
The proof of Lemma 3 will construct (by ind on the depth) a K-evaluation using some very specific kind of CMDT, i.e. canonical trees. To keep their depth small we will use restrictions and a Switching Lemma then since K-evaluations are well-behaved under restrictions then we will be able to build a K-evaluation in Lemma 3. This is from a very high level perspective the plan of the lecture.

- K-evaluations are well-behaved under restrictions -

Given a tree $T \in CMDT_n$ and $\alpha \in \mathcal{M}_n$, we define $T\restriction_\alpha \in CMDT_n$

(i) if $|T| = 1$, i.e. the tree has just one node, then $T\restriction_\alpha = T$;

(ii) if $T =$  ; if i is not touched by α then

$T\restriction_\alpha =$ 

- if $(i,j) \in \alpha$ for some j, then some of the edges going out from i has label (i,j) and let T_j its sub-tree. then $T\restriction_\alpha = T_j\restriction_\alpha$.

observation 3: let $\nu : \Gamma \longrightarrow CMDT_n$ a K-evaluation of a set of formulas Γ and let $\alpha \in \mathcal{M}_n$. Then $\nu' : \Gamma\restriction_\alpha \longrightarrow CMDT(V\restriction_\alpha)$ defined as follows is a K-evaluation.

$\nu'(0)$ is the tree with a single node with label 0, same for $\nu'(1)$ and label 1

For a non-trivial $F\restriction_\alpha \in \Gamma\restriction_\alpha$ with $F \in \Gamma$ we set $\nu'(F\restriction_\alpha) = \nu(F)\restriction_\alpha$.

proof sketch: $\nu'(x_{ij})$ has the correct form (check).

given a tree $T \in CMDT_n$ let T^c be the same tree as T but with labels of the leaves exanged from 0 to 1 and viceversa.

$$\nu'(\neg F\restriction_\alpha) = \nu(\neg F)\restriction_\alpha = \nu(F)^c\restriction_\alpha \underset{\text{exercise}}{=} (\nu(F)\restriction_\alpha)^c = \nu'(F\restriction_\alpha)^c.$$

If $F \in \Gamma$ is a disj with merged form $\bigvee_{i\in I} F_i$, by hyp. $\nu(F)$ represents $\bigvee_{i\in I} Disj(\nu(F_i))$, then (exercise) $\nu(F)\restriction_\alpha$ represents $\bigvee_{j\in I} Disj(\nu(F_i))\restriction_\alpha$

$$\underset{\text{exercise}}{\equiv} \bigvee_{i\in I} Disj(\nu(F_i)\restriction_\alpha).$$

So $\nu'(F\restriction_\alpha)$ represents $\bigvee_{i\in I} Disj(\nu'(F\restriction_\alpha))$.

$\square$

Lemma 3 (restated): Let d be an integer, $0<\varepsilon<\frac{1}{5}$, $0<\delta<\varepsilon^d$ and Γ a set of formulas of depth $\leq d$ closed under subformulas.
If $|\Gamma|<2^{n^\delta}$ then $\exists \rho \in M_n^q$ with $q=n^{\varepsilon^d}$ s.t. there is a $2n^\delta$-evaluation of $\Gamma\restriction_\rho$

proof: By induction on d.

x and ¬x have depth 0

$d=0$: Γ just contains constants and variables and negation of variables.
By construction then there is a 2-evaluation of Γ. We can just set $\rho=\emptyset$.

"$d-1 \rightsquigarrow d$": Let Γ be a set of formulas of depth d closed under subformulas, $|\Gamma|<2^{n^\delta}$ with $0<\delta<\varepsilon^d$.
Let $\Gamma'\subseteq\Gamma$ be the set of all the subformulas of Γ of depth $\leq d-1$, since $0<\delta<\varepsilon^d<\varepsilon^{d-1}$ then by the ind. hyp. there exists a $\rho\in M_n^q$ with $q=n^{\varepsilon^{d-1}}$ and a $2n^\delta$-evaluation v of $\Gamma'\restriction_\rho$.
Let F be a disj in $\Gamma\restriction_\rho$ of depth d with merged form $\bigvee_{i\in I} F_i$ (so in particular all $F_i\in\Gamma'\restriction_\rho$). Let $D_F=\bigvee_{i\in I}\mathrm{Disj}(v(F_i))$, so in part. D_F is a $2n^\delta$-matching DNF. Let $q'=n^{\varepsilon^d}$ and

$$\mathrm{Bad}_q^{q'}(D_F,2n^\delta)=\{\rho'\in M_q^{q'}: \text{depth of } T(D_F,\rho') \text{ is } \geq 2n^\delta\},$$

then by the Switching Lemma:

$$\frac{|\mathrm{Bad}_q^{q'}(D_F,2n^\delta)|}{|M_q^{q'}|}\leq\left(\frac{2(2n^\delta)(2n^{\varepsilon^d}+1)^4}{n^{\varepsilon^{d-1}}-n^{\varepsilon^d}}\right)^{n^\delta}\overset{(*)}{\leq}2^{-n^\delta}. \quad (\dagger)$$

The inequality $(*)$ holds for n large enough since

$$\frac{2(2n^\delta)(2n^{\varepsilon^d}+1)^4}{n^{\varepsilon^{d-1}}-n^{\varepsilon^d}}\approx c\cdot n^{\delta+4\varepsilon^d-\varepsilon^{d-1}}\xrightarrow[n\to\infty]{}0$$

as long as $\varepsilon<\frac{1}{5}$: $\varepsilon^d<\varepsilon^{d-1}/5$, $\delta<\frac{\varepsilon^{d-1}}{5}$.

Since $|\Gamma|<2^{n^\delta}$ then clearly $|\Gamma\restriction_\rho|<2^{n^\delta}$ and by $(\dagger)$ there must exist some $\rho'\in M_q^{q'}$ s.t. for every F disj in $\Gamma\restriction_\rho$ of depth d, $T(D_F,\rho')$ has depth $\leq 2n^\delta$.

Now $\rho \cup \rho' \in M_n^{q'}$ by construction so it is just remained to build a $2n^\delta$-evaluation v' for $\Gamma\restriction_{\rho\cup\rho'}$. We do that as follows:

$$v'(F\restriction_{\rho\cup\rho'}) = \begin{cases} v(F\restriction_\rho)\restriction_{\rho'} & \text{if } F\restriction_\rho \text{ has depth} \leq d \\ v(G\restriction_\rho)\restriction_{\rho'}^c & \text{if } F\restriction_\rho \text{ has depth } d \text{ and } F = \neg G \\ T(D_{F\restriction_\rho}, \rho') & \text{if } F\restriction_\rho \text{ has depth } d \text{ and } F \text{ is a disj} \\ & \text{with merged form } \bigvee_{i\in I} F_i \text{ and} \\ & D_{F\restriction_\rho} = \bigvee_{i\in I} \mathrm{Disj}(v(F_i\restriction_\rho)) \end{cases}$$

<u>exercise</u>: check that v' is a K-evaluation of $\Gamma\restriction_{\rho\cup\rho'}$.

(This follows basically from the fact that matching trees, $\mathrm{Disj}(\cdot)$ and the notion of "represents" are well-behaved under restrictions.

$\square$