DD2445 COMPLEXITY THEORY

L3 I

LECTURE 3

Last time:

Polynomial-time computation [P]

Efficient computation and Church-Turing thesis

Reductions
- solve problems
- relate hardness of problems

Polynomial-time verifiability [NP]

Nondeterministic Turing machines

Gave examples of problems in NP
Mentioned that some of them are "the hardest":

NP-complete

Let us define what that means

DEF 6 $L \subseteq \{0,1\}^*$ is *polynomial-time Karp reducible* to $L' \subseteq \{0,1\}^*$, denoted $L \leq_p L'$, if $\exists$ poly-time computable function $f: \{0,1\}^* \to \{0,1\}^*$ s.t. $\forall x \quad x \in L$ iff $f(x) \in L'$

DEF 7 L' is *NP-hard* if $L \leq_p L'$ $\forall L \in NP$

L' is *NP-complete* if in addition $L' \in NP$

> L' is as hard as any problem in NP, since any algorithm [efficient] for L' can be used to decide any language in NP efficiently.

LEMMA 8

1. If $L \leq_p L'$ and $L' \leq_p L''$ then $L \leq_p L''$ (TRANSITIVITY)
2. If L is NP-hard and $L \in P$, then $P = NP$.
3. If L is NP-complete, then $L \in P$ iff $P = NP$

Proof See textbook.

Sooo... are there NP-complete problems? Yes, tons of them. In fact, all over the place in math, CS, physics, chemistry, biology, economics, industry... [But this course focuses on *theory* not applications.]

The most important (?) ones: Variants of SAT

CNF formula (conjunctive normal form)

Variables x, y, z set to true = 1 or false = 0

Logical connectives AND $\wedge$, OR $\vee$, NOT $\neg$

Except for now negate only variables, written $\overline{x}$

Literal x or $\overline{x}$ True if $x=1$ or $x=0$

(Disjunctive) clause $C = x \vee \overline{y} \vee z$ denoted F or φ, usually

satisfied ~~True~~ if one literal true

CNF formula = conjunction of clauses

$C_1 \wedge C_2 \wedge \cdots \wedge C_m = \bigwedge_{i=1}^{m} C_i$

satisfied if all clauses satisfied.

CNFSAT (just SAT in Arora-Barak)

Given CNF formula F, does it have a satisfying assignment?

Two variants

3-SAT: Each clause has size at most 3

E3-SAT: Each clause has size exactly 3

[Notation varies]

COOK-LEVIN THEOREM ('71 and '73, resp)

1. CNFSAT is NP-complete.
2. 3-SAT is NP-complete.

Satisfying assignment clearly easy to verify. Need to show every other $L \in NP$ reduces to CNFSAT.

What can we do?

- Only thing we know is that $\exists$ TM that verifies witnesses to claim $x \in L$ and accepts if correct.
- Write computation of such TM on x as ~~Bo~~ CNF formula
- Show that can plug in witness so that TM computation is satisfying.

PROP 9 Any Boolean function $f: \{0,1\}^\ell \to \{0,1\}$ can be expressed as a CNF of size $\ell \cdot 2^\ell$

(size = # connectives $\wedge/\vee$ or total # literals)

Look at all assignments α s.t. $f(\alpha) = 0$

Suppose $\alpha = \langle 1, 1, 0, 1 \rangle$

Write down clause ruling out this assignment $C_\alpha = \bar{x}_1 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_4$

Then $F = \bigwedge_{\alpha \in f^{-1}(0)} C_\alpha$ represents f

(But is of exp size)

Want to construct reduction $x \to F_x$ s.t.

$$F_x \in SAT \iff \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

Apply Prop 9 $\Rightarrow$ gives formula of size

$$p(|x|) \cdot 2^{p(|x|)}$$

Use that TM does <u>local</u> computation

Two simplifying (but justifiable) assumptions:

(1) M has two tapes, input and work/output
(2) M is <u>oblivious</u>: head movements don't depend on tape contents, only on input length $|x|$

At most quadratic loss in running time — see AB Ch 1

Can run M on x and $0^{p(|x|)}$ to determine head positions at every time step.

Q: set of states of TM (= lines in program)
Γ: alphabet (including 0, 1, _Kblank) etc)

y = x and u concatenated

SNAPSHOT $z = \langle a, b, q \rangle \in \Gamma \times \Gamma \times Q$

a, b symbols read from tapes
q current state

z can be encoded as binary string of fixed length say c bits

Snapshot at time i depends on
(a) state at time $i-1$
(b) contents of current locations of tapes.

Suppose we're given sequence of snapshots
$z_1, z_2, z_3, \ldots, z_t$ claimed to be correct computation. How to verify?

To check z_i, only need to look at

(1) z_{i-1} – did we jump to current state?

(2) $y_{inputpos(i)}$ – The input tape head is at position $inputpos(i)$, which we have computed – is the symbol in the snapshot consistent with this

(3) $z_{prev(i)}$ – $prev(i)$ was the last time current pos of work tape was visited. Looking at $z_{prev(i)}$ we can see what symbol was written to work tape then. Is it consistent).

(1)-(3) uniquely determine z_i

So $\exists$ function $f(z_i, z_{i-1}, y_{inputpos(i)}, z_{prev(i)})$ that evaluates to true when transition correct

function of $c + c + 1 + c$ bits = constant

Apply Prop 9 $\Rightarrow$ constant-size formula

Write down CNF formula F_x saying

(1) First $|x|$ bits on input tape are x
(2) z_1 encodes starting position of TM
(3) For every $i > 1$, z_i is correct transition given z_{i-1}, $y_{inputpos(i)}$, $z_{prev(i)}$.
(4) The last snapshot $z_{T(n)}$ is that of an accepting computation (1 is on the work tape).

Size of formula : $T(n) \cdot$ (exponential in c)

Can be computed in poly time
- First simulate $M(x, 0^{P(|x|)})$ to compute positions.
- Then output CNF encoding transitions for $T(n)$ steps and correct conditions at start & stop.

Reducing from CNFSAT to 3-SAT

Straightforward exercise — see textbook.

Two observations

(1) Formula F_x (and time to compute it) can be made very small — $O(T \log T)$

(2) From satisfying assignment to F_x, can read off witness u for x

Levin reduction

Now to prove NP-completeness of L, sufficient to reduce from CNFSAT / 3-SAT.

And then to prove L' NP-complete, reduce from CNFSAT, 3-SAT, or L. Et cetera...

(And, in fact, on the positive side, many practical problems can be solved by reducing to CNFSAT and then solved if you have a good SAT solver — more about that later).

Reductions are all over complexity theory and have proven to be an extremely useful tool (and in TCS in general).

THM 10 ~~LINEARPROGRAM~~ 0/1-INTEGERPROGRAMMING is NP-complete.

Proof Easy to verify suggested solution.

Reduction: Translate clauses to lin ineq

$$x \to x$$
$$\bar{x} \to 1 - x$$

$$x \vee \bar{y} \vee z \longrightarrow x + (1-y) + z \geq 1$$

This lin ineq satisfied by 0/1-assignment exactly when clause satisfied

3-COLOURING

Graph $G=(V,E)$. Colour vertices red/blue/green so that if $(u,v) \in E$ then u and v have different colours.

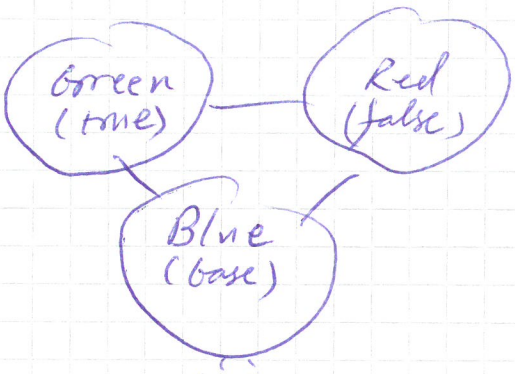
THEOREM 11

3-COLOURING is NP-complete

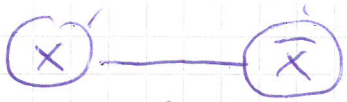
Proof Easy to verify a colouring.

Reduce from 3-SAT

Triangle

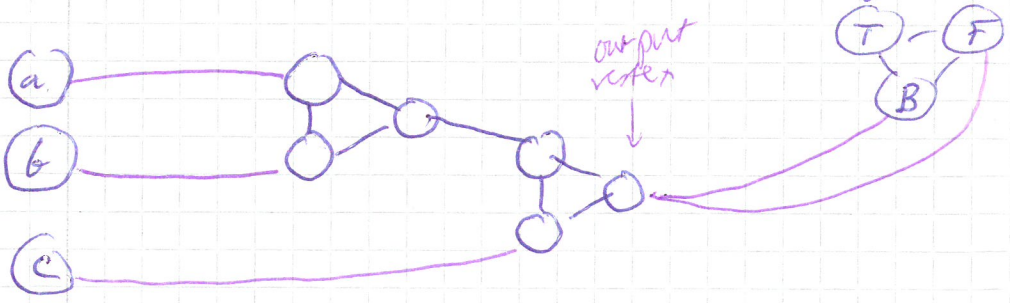


For each variable, x.



connect both x and $\bar{x}$ to Blue (base)

For clause $a \vee b \vee c$ vertices a, b, c already exist.



Turns F into graph G_F (L23 X)

Need to prove

F SAT $\Rightarrow$ G 3-colourable

G 3-colourable $\Rightarrow$ F SAT

($\Rightarrow$) Colour true literals green
Colour false literals red
Then output vertex of clause gadget can always be coloured Green/True if
Clause contains true literal

($\Leftarrow$) Given a colouring, identify
T-colour with true
F-colour with false

We get some truth value assignment since x and $\bar{x}$ have opposite values.

Every clause gadget has output vertex coloured true

Only possible if some literal in clause true.

Decision vs search

THEOREM

Suppose $P = NP$. Then for every NP-language L with verifier M can find poly-time TM B that finds certificate for x (i.e., solves x).

Proof sketch:

Start with CNFSAT

Given $F = F(x_1, \ldots, x_n)$

Set $x = 1$, simplify, check if $F|_{x=1}$ satisfiable

If not check if $F|_{x=0}$ satisfiable

Fix $x = b$ so that $F|_{x=b}$ sat, and now move on to x_2

Will yield satisfying assignment

For general L, reduce $x \to F_x$,

find sat assignment for F_x, read off a.

CNFSAT is <u>downward self-reducible</u>

Given an efficient algorithm that solves CNFSAT on input size $< n$, can solve CNFSAT instances of size n.

All NP-complete problems have similar property (follows from Cook-Levin)

For $L \subseteq \{0,1\}^*$ language, the complement of L is $\overline{L} = \{0,1\}^* \setminus L$

DEF $\quad coNP = \{ L \mid \overline{L} \in NP \}$

Aside: If strings in L encode objects such as e.g. graphs, then we usually think of $\overline{L}$ as only containing correctly encoded instances.
I.e. L and $\overline{L}$ will both be sets of graphs satisfying and not satisfying some property respectively, while "garbage strings" are not contained in either L or $\overline{L}$.
Not an issue, really

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coNP not the complement of NP

Intersection non-empty
E.g. $P \subseteq NP \cap coNP$ (why?)

Example CNFUNSAT $\in$ coNP

In fact, CNFUNSAT is coNP-complete — any other coNP-language is reducible to it. Given $L \in coNP$, run book-keeper on $\overline{L}$. $x \in L \Leftrightarrow x \notin \overline{L} \Leftrightarrow F_x \notin CNFSAT \Leftrightarrow F_x \in CNFUNSAT$

How is coNP related to NP?
Could there be short certificates that none of exponentially many assignments satisfy a formula?

Most researchers believe $NP \neq coNP$
We don't know though.

Nondeterministic exponential time

DEF $NEXP = \bigcup_{c \in \mathbb{N}} NTIME(2^{n^c})$

clearly $P \subseteq NP \subseteq EXP \subseteq NEXP$

Why care about such classes?

THEOREM If $EXP \neq NEXP$, then $P \neq NP$

Proof Contrapositive: suppose $P = NP$, prove $EXP = NEXP$

Suppose $L \in NTIME(2^{n^c})$; NDTM M decides L.

Consider $L_{pad} = \{ \langle x, 1^{2^{|x|^c}} \rangle : x \in L \}$

Claim $L_{pad} \in NP$. First check that input is in correct form then run M. Since input exp large, this is poly time. By assumption $L_{pad} \in P$.
But then $L \in EXP$. Namely given x, pad it with ones and then check whether new string is in L_{pad}.

PADDING Useful technique to "scale up" or "scale down" results between weaker and stronger complexity classes

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What does all of this mean?

Sec 2.7 in Arora-Barak highly recommended reading

Don't have time to discuss — our time is up

TODAY Focus on NP and NP-completeness

Should have seen most of this before

Though probably not presented in this way

FUTURE LECTURES

- New stuff (probably)
- Can we separate cplx classes at all? Is P distinct from EXP?
- Can we say anything about landscape between P and NP?

But first:

A detour into CIRCUIT COMPLEXITY

Lecture on Friday +

2 guest lectures with

JOHAN HÅSTAD