LECTURE 2

TODAY: NP, NP-completeness, Cook-Levin theorem, reductions, coNP, EXP, NEXP

Solving vs verifying

Doing an exam requires coming up with solutions
— can be hard

Grading just involves verifying correctness
— much easier (hopefully... usually...)

Last time: Complexity class P
Polynomial-time computation
Efficiently solvable problems

Today: Complexity class NP
Problem for which solutions can be verified efficiently

DEF 1 Language L is in NP if $\exists$ polynomial p and poly-time TM M (verifier) s.t.

$$x \in L \iff \exists u \in \{0,1\}^{p(|x|)} \text{ s.t. } M(x,u) = 1$$

u is a certificate or witness for x

EXAMPLES

① TRAVELLING SALESMAN

n cities, $\binom{n}{2}$ pairwise distances d_{ij}
length constraint k
Is there a tour visiting every city exactly one and having length $\leq k$

② SUBSET SUM

n numbers $A_1, \dots, A_n$ and number T,
is there subset $S \subseteq [n]$ s.t. $\sum_{i \in S} A_i = T$

③ LINEAR PROGRAMMING

m linear inequalities $a_1 u_1 + a_2 u_2 + \dots + a_n u_n \leq b$
$a_i, b \in \mathbb{Q}$, is there an assignment to the u_i satisfying all inequalities?

④ 0/1 INTEGER PROGRAMMING

Same as above, but assignments to u_i in $\{0,1\}$

⑤ GRAPH ISOMORPHISM

Given two graphs G_1, G_2, are they isomorphic? I.e., Is there a bijection $\pi : V(G_1) \to V(G_2)$ s.t. $(u,v) \in E(G_1)$ iff $(\pi(u), \pi(v)) \in E(G_2)$

⑥ COMPOSITE NUMBERS

Given $N \in \mathbb{N}$, decide if N is composite (i.e., not a prime)

(7) FACTORING

Given $N, \ell, u \in \mathbb{N}$, does N have a prime factor p with $\ell \leq p \leq u$

(8) CONNECTIVITY

Given graph G and vertices s, t, are s and t connected in G

(9) CNF UNSAT

Given a CNF formula $F = \bigwedge_{i=1}^{m} C_i$ for clauses C_i on the form $x_1 \vee \overline{x_2} \vee \overline{x_3} \vee x_4$, does every truth value assignment fail to satisfy at least one clause?

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(1) TSP: witness: tour
complexity: hardest problem in NP: NP-complete

(2) witness: subset
complexity: NP-complete

(3) witness: assignment
complexity: For long best alg $\overset{\text{worst-case}}{\text{exptime}}$ Simplex
Khachiyan '79: $\in P$
Karmarkar '84: Efficient

(4) witness: assignment, complexity NP-complete

(5) witness: bijection cplx: not believed to be in P
— 11 — NP-complete

(6) witness: factor
known to be solvable with randomness
[Miller '76, Rabin '80]
In P! [AKS '04]

(7) witness: prime factor
not believed to be in P (RSA breaks down)
not believed to be NP-complete

(8) witness: path cplx in P (do BFS)

(9) witness: ?
in coNP – see later

PROPOSITION 3 $P \subseteq NP \subseteq EXP$

DEF 2 EXP (or EXPTIME) $= \bigcup_{c=1}^{\infty} \mathrm{DTIME}(2^{n^c})$

Proof $P \subseteq NP$: pick witnesses of length 0.
$NP \subseteq EXP$: At most exp many witness candidates, try all in poly-time per candidate.

Prop 2 is state-of the art (sadly).
One of the Millennium Problems: $P \stackrel{?}{=} NP$
Most (but not all) believe $P \neq NP$.

Original definition of NP
uses nondeterminism (the "N" in "NP")

Nondeterministic TM

Each line in program has two variants
(TM has two transition functions)
At each step, TM arbitrarily chooses one
Think of it as flipping a "golden coin" that
always comes up "the best way"

NTM
NDTM accepts x if at least one possible sequence of choices leads to accept; otherwise rejects

NDTM/NTM runs in time $T(n)$ if all possible sequences of choices terminate within time $T(n)$.

Aside: accept means either
(a) write 1 on output tape, or
(b) reach special state q_{accept}

DEF 4 L is in $NTIME(T(n))$ if $\exists\, c > 0$ and $c \cdot T(n)$ - time NDTM M such that for every $x \in \{0,1\}^*$ $x \in L$ iff $M(x) = 1$

LEMMA 5 $NP = \bigcup_{c \in \mathbb{N}} NTIME(n^c)$

Proof Need to prove

(1) $L \in NP \Rightarrow L \in \bigcup_{c \in \mathbb{N}} NTIME(n^c)$

(2) $L \in$ ~~NTIME~~ $\bigcup_{c \in \mathbb{N}} NTIME(n^c) \Rightarrow L \in NP$

(1) There is some witness. Let NDTM write down a witness nondeterministically, then verify (by running verifier for L)

$x \in L \Rightarrow \exists$ good witness $\Rightarrow$ accept

$x \notin L \Rightarrow$ no good witness $\Rightarrow$ reject

Runs in poly-time, since verifier poly-time and witness poly-length.

(2) Let the witness be a good sequence of "golden coin tosses". Simulate NDTM with this sequence to check acceptance.

Wouldn't it be great to have an NDTM on your desk?

Not physically realizable (as far as we know)

But useful theoretical model, e.g. for exhaustive search

DEF 6 $L \subseteq \{0,1\}^*$ is <u>polynomial-time Karp reducible</u> to $L' \subseteq \{0,1\}^*$, denoted $L \leq_p L'$, if $\exists$ poly-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ s.t. $\forall x$ $x \in L$ iff $f(x) \in L'$

DEF 7 L' is <u>NP-hard</u> if $L \leq_p L'$ $\forall L \in NP$

L' is <u>NP-complete</u> if in addition $L' \in NP$

> L' is as hard as any problem in NP, since any efficient algorithm for L' can be used to decide any language in NP efficiently.

LEMMA 8

1. If $L \leq_p L'$ and $L' \leq_p L''$ then $L \leq_p L''$ (TRANSITIVITY)
2. If L is NP-hard and $L \in P$, then $P = NP$.
3. If L is NP-complete, then $L \in P$ iff $P = NP$

Proof See textbook.

Sooo... Are there NP-complete problems? Yes, tons of them. In fact, all over the place in math, CS, physics, chemistry, biology, economics, industry... [But this course focuses on <u>theory</u> not applications.]

The most important (?) ones: Variants of SAT

CNF formula (conjunctive normal form)

Variables x, y, z set to true $= 1$ or false $= 0$

Logical connectives AND $\wedge$, OR $\vee$, NOT $\neg$

Except for now negate only variables, written $\overline{x}$

Literal x or $\overline{x}$ True if $x = 1$ or $x = 0$

(Disjunctive) clause $C = x \vee \overline{y} \vee z$

Satisfied ~~True~~ if one literal true

denoted F or φ, usually

CNF formula $\triangleq$ conjunction of clauses

$$C_1 \wedge C_2 \wedge \cdots \wedge C_m = \bigwedge_{i=1}^{m} C_i$$

Satisfied if all clauses satisfied.

CNFSAT (just SAT in Arora-Barak)

Given CNF formula F, does it have a satisfying assignment?

Two variants

3-SAT: Each clause has size at most 3

E3-SAT: Each clause has size exactly 3

[Notation varies]

COOK-LEVIN THEOREM ('71 and '73, resp)

1. CNFSAT is NP-complete.
2. 3-SAT is NP-complete.

Satisfying assignment clearly easy to verify. Need to show every other $L \in NP$ reduces to CNFSAT.

What can we do?

- Only thing we know is that $\exists$ TM that verifies witnesses to claim $x \in L$ and accepts if correct.
- Write computation of such TM on x as ~~Bo~~ CNF formula
- Show that can plug in witness so that TM computation is satisfying.

PROP 9 Any Boolean function $f : \{0,1\}^\ell \to \{0,1\}$ can be expressed as a CNF of size $\ell \cdot 2^\ell$

(size = # connectives $\wedge / \vee$ or total # literals)

Look at all assignments α s.t. $f(\alpha) = 0$

Suppose $\alpha = \langle 1, 1, 0, 1 \rangle$

Write down clause ruling out this assignment $C_\alpha = \overline{x}_1 \vee \overline{x}_2 \vee x_3 \vee \overline{x}_4$

Then $F = \bigwedge_{\alpha \in f^{-1}(0)} C_\alpha$ represents f.

(But is of exp size)

Want to construct reduction $x \to F_x$ s.t.

$F_x \in SAT \iff \exists u \in \{0,1\}^{p(|x|)}$ s.t. $M(x,u)=1$

Apply Prop 9 $\Rightarrow$ gives formula of size

$$p(|x|) \cdot 2^{p(|x|)}$$

Use that TM does <u>local</u> computation

Two simplifying (but justifiable) assumptions:

1. M has two tapes, input and work/output
2. M is <u>oblivious</u>: head movements don't depend on tape contents, only on input length $|x|$

At most quadratic loss in running time — see AB Ch 1

Can run M on x and $0^{p(|x|)}$ to determine head positions at every time step.

Q: set of states of TM (= lines in program)
Γ: alphabet (including 0, 1, <blank> etc)

y = x and u concatenated

SNAPSHOT $z = \langle a, b, q \rangle \in \Gamma \times \Gamma \times Q$

a, b symbols read from tapes
q current state

z can be encoded as binary string of fixed length say c bits

Snapshot at time i depends on
(a) state at time $i-1$
(b) contents of current locations of tapes.

Suppose we're given sequence of snapshots
$z_1, z_2, z_3, \ldots, z_t$ claimed to be correct computation. How to verify?

To check z_i, only need to look at

① z_{i-1} — did we jump to current state?

② $y_{\text{inputpos}(i)}$ — The input tape head is at position inputpos(i), which we have computed — is the symbol in the snapshot consistent with this

③ $z_{\text{prev}(i)}$ — prev(i) was the last time current pos of work tape was visited. Looking at $z_{\text{prev}(i)}$ we can see what symbol was written to work tape then. (Is it consistent).

①-③ uniquely determine z_i

So $\exists$ function $f(z_i, z_{i-1}, y_{\text{inputpos}(i)}, z_{\text{prev}(i)})$ that evaluates to true when transition correct

function of $c + c + 1 + c$ bits $=$ constant

Apply Prop 9 $\Rightarrow$ constant-size formula

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Write down CNF formula F_x saying

1. First $|x|$ bits on input tape are x
2. z_1 encodes starting position of TM
3. For every $i > 1$, z_i is correct transition given $z_{i-1}, y_{inputpos(i)}, z_{prev(i)}$.
4. The last snapshot $z_{T(n)}$ is that of an accepting computation (1 is on the worktape).

Size of formula : $T(n) \cdot$ (exponential in c)

Can be computed in poly time
- First simulate $M(x, 0^{p(|x|)})$ to compute positions.
- Then output CNF encoding transitions for $T(n)$ steps and correct conditions at start & stop.

Reducing from CNFSAT to 3-SAT

Straightforward exercise — see textbook.

Two observations

1. Formula F_x (and time to compute it) can be made very small — $O(T \log T)$
2. From satisfying assignment to F_x, can read off witness u for x

Levin reduction

Now to prove NP-completeness of L, sufficient to reduce from CNFSAT / 3-SAT.

And then to prove L' NP-complete, reduce from CNFSAT, 3-SAT, or L. Etcetera...

(And, in fact, on the positive side, many practical problems can be solved by reducing to CNFSAT and then solved if you have a good SAT solver — more about that later).

Reductions are all over complexity theory and have proven to be an extremely useful tool (and in TCS in general).

THM 10 ~~LINEAR PROGRAM~~ 0/1-INTEGER PROGRAMMING is NP-complete.

Proof Easy to verify suggested solution.

Reduction: Translate clauses to lin ineq

$$x \to x$$
$$\overline{x} \to 1 - x$$

$$x \vee \overline{y} \vee z \longrightarrow x + (1-y) + z \geq 1$$

This lin ineq satisfied by 0/1-assignment exactly when clause satisfied

3-COLOURING

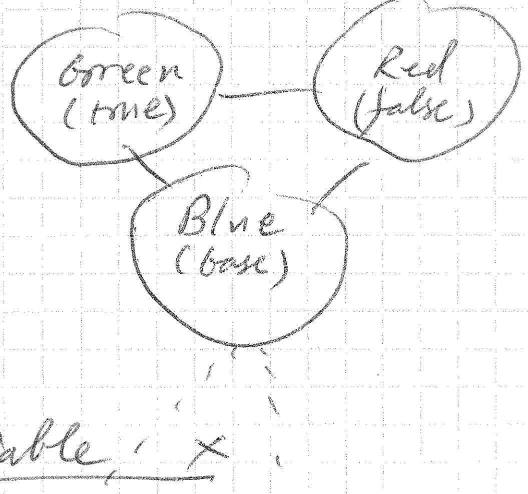
Graph $G = (V, E)$. Colour vertices red/blue/green so that if $(u,v) \in E$ then u and v have different colours.

THEOREM 11

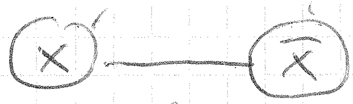
3-COLOURING is NP-complete

Proof Easy to verify a colouring.
Reduce from 3-SAT

Triangle

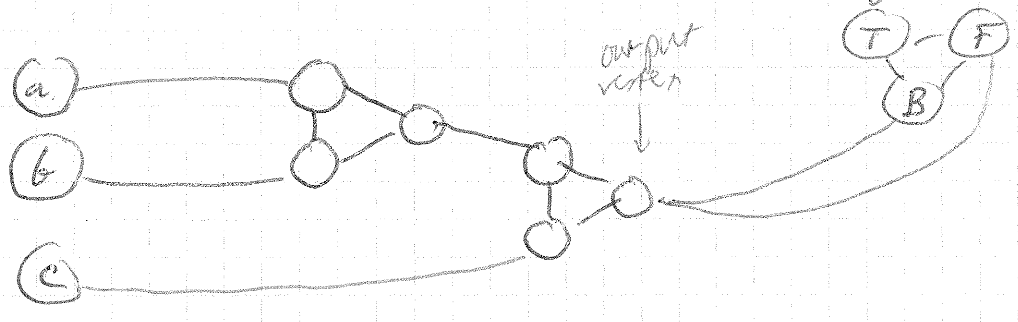


For each variable: x



connect both x and $\bar{x}$ to Blue (base)

For clause $a \vee b \vee c$ vertices a, b, c already exist.



Turns F into graph G_F L2 XV

Need to prove

F SAT $\Longrightarrow$ G 3-colourable

G 3-colourable $\Rightarrow$ F SAT

($\Rightarrow$) Colour true literals green
Colour false literals red
Then output vertex of clause gadget can always be coloured Green/True if clause contains true literal

($\Leftarrow$) Given a colouring, identify
T-colour with true
F-colour with false

We get some truth value assignment since x and $\bar{x}$ have opposite values.

Every clause gadget has output vertex coloured <u>true</u>

Only possible if some literal in clause true.

Decision vs search

THEOREM
Suppose P=NP. Then for every NP-language L with verifier M can find poly-time TM B that finds certificate for x (i.e., solves x).

Proof sketch:
Start with CNFSAT
Given $F = F(x_1, \ldots, x_n)$
Set $x = 1$, simplify, check if $F\restriction_{x=1}$ satisfiable
If not check if $F\restriction_{x=0}$ satisfiable
Fix $x = b$ so that $F\restriction_{x=b}$ sat, and now move on to x_2
Will yield satisfying assignment
For general L, reduce $x \to F_x$, find sat assignment for F_x, read off a.

CNFSAT is downward self-reducible

Given an efficient algorithm that solves CNFSAT on input size $< n$, can solve CNFSAT instances of size n.

All NP-complete problems have similar property (follows from Cook-Levin)

For $L \subseteq \{0,1\}^*$ language, the <u>complement</u> of L is $\overline{L} = \{0,1\}^* \setminus L$

<u>DEF</u> $coNP = \{ L \mid \overline{L} \in NP \}$

Aside: If strings in L encode objects such as e.g. graphs, then we usually think of $\overline{L}$ as only containing <u>correctly encoded</u> instances. I.e. L and $\overline{L}$ will both be sets of graphs satisfying and not satisfying same property respectively, while "garbage strings" are not contained in either L or $\overline{L}$.
Not an issue, really

coNP <u>not</u> the complement of NP

Intersection non-empty
E.g. $P \subseteq NP \cap coNP$ (why?)

<u>Example</u> CNFUNSAT $\in$ coNP

In fact, CNFUNSAT is coNP-complete — any other coNP-language is reducible to it. Given $L \in coNP$, run Cook-Levin on $\overline{L}$. $x \in L \Leftrightarrow x \notin \overline{L} \Leftrightarrow F_x \notin CNFSAT$
$\Leftrightarrow F_x \in CNFUNSAT$

How is coNP related to NP?
Could there be short certificates that none of exponentially many assignments satisfy a formula?

Most researchers believe NP $\neq$ coNP
We don't know, though.

Nondeterministic exponential time

DEF NEXP $= \bigcup_{c \in \mathbb{N}} NTIME(2^{n^c})$

clearly $P \subseteq NP \subseteq EXP \subseteq NEXP$

Why care about such classes?

THEOREM If EXP $\neq$ NEXP, then $P \neq NP$

Proof Contrapositive: suppose $P = NP$, prove EXP = NEXP

Suppose $L \in NTIME(2^{n^c})$; NDTM M decides L.

Consider $L_{pad} = \{ \langle x, 1^{2^{|x|^c}} \rangle : x \in L \}$

Claim
$L_{pad} \in NP$. First check that input is on correct form then run M. Since input exp large, this is poly time. By assumption $L_{pad} \in P$.
But then $L \in EXP$. Namely given x, pad it with ones and then check whether new string is in L_{pad}.

<u>Padding</u> useful technique to "scale up" or "scale down" results between weaker and stronger complexity classes.

What does all of this mean?

See 2.7 in Arora-Barak highly recommended reading.

Don't have time to discuss — our time is up.

<u>TODAY</u>: Focus on NP and NP-completeness. Should have seen most of this before, although probably not presented in this way

<u>NEXT TIME</u> — New stuff (probably)

- Can we separate cplx classes at all? Is P distinct from EXP?!
- Can we say anything about the landscape between P and NP

Might be good to skim Chap 3 before. This is hard stuff and so sometimes tricky to absorb online