DD2446 kplx13:

PCP Theorem & Stuff

- ☐ Constraint Satisfaction, Approximability
- ☐ PCP Theorem in its two forms
- ☐ Proof overview of PCP Thm
- (☐ Parallel Repetition, Label Cover)

Max k-CSP_q

* Instance φ consists of n variables $x_1, \dots, x_n$ over domain $[q] = \{1, \dots, q\}$, m constraints $(I_1, C_1), \dots, (I_m, C_m)$ where $I_j = \{i_{j,1}, i_{j,2}, \dots, i_{j,k}\}$ is a set of k variable indices and $C_j : [q]^k \to \{0,1\}$ is a predicate on k variables.

* Assignment α satisfies (I_j, C_j) if $C_j(\alpha_{i_{j1}}, \alpha_{i_{j2}}, \dots, \alpha_{i_{j,k}}) = 1$ $[= C_j(\alpha_{I_j})]$

* Goal: find assignment α s.t. number of satisfied constraints is maximized

<u>Examples:</u> * MAX k-SAT is a special case of Max k-CSP$_q$

* "MAX 3-Coloring" is a special case of Max ~~2~~-CSP$_3$ | (color vertices of a graph with 3 colors s.t. as many edges as possible ~~have~~ different colors on the endpoints)

<u>Def</u> (Optimum) value of φ:

$\mathrm{val}(\varphi) \in [0,1]$, maximum fraction of constraints satisfied by any assignment.

<u>E.g.</u> $\mathrm{val}(\varphi) = 1$: can satisfy all constraints

$\mathrm{val}(\varphi) = 0$: can not satisfy any constraints.
(can this happen?)

<u>Approximation</u>

<u>Note:</u> Max k-CSP$_q$ is NP-hard for most values of k, q. (When is it in P?)

But maybe we can find approximately optimal solutions?

Def Algorithm A is an α-approximation algorithm ~~that~~ if it outputs an assignment s.t.

$$\mathrm{val}(A(\varphi);\varphi) \geq \alpha \cdot \mathrm{val}(\varphi)$$

(i.e. within factor α of optimal value)

Long history, by no means restricted to CSPs

More: Schmoys-Williamson book (it's free!)

PCP Theorem (version 1)

There is a universal constant $\delta > 0$ s.t. given a MAX 3-SAT instance φ, it is NP-hard to distinguish between

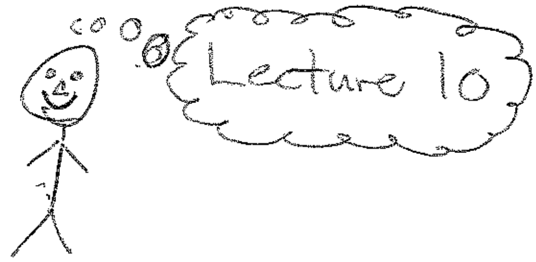
$$\mathrm{Opt}(\varphi) \geq 1 \quad \text{and} \quad \mathrm{Opt}(\varphi) \leq 1-\delta$$

$$\Downarrow$$

MAX 3-SAT is NP-hard to approximate within a factor $1-\delta$!

(Why not equivalence?)

PCP?



<u>Recall / Def</u> PCP verifier V for language L:

gets ~~x~~ instance $x \overset{?}{\in} L$ and a "proof" π that $x \in L$.

- V flips a few random coins, then looks at a few positions in π, and either accepts or rejects the proof. And runs in poly-time

<u>Two key properties:</u>

- Completeness c: if $x \in L$ then $\exists$ proof π such that
$$\Pr_{\substack{\text{randomness} \\ \text{of } V}}[V \text{ accepts}] \geq c$$

- Soundness s: if $x \notin L$ then
$$\forall \text{ "proofs" } \pi : \Pr[V \text{ accepts}] \leq s$$

Def $PCP_{c,s}[r, q, \Sigma]$ = class of languages having a PCP verifier that uses r random bits, makes q queries, proof alphabet size Σ, completeness c,

Examples:

$$PCP_{1,0}[0, poly(n), 2] = NP \quad \text{(by definition!)}$$

$$PCP_{1,0}[0, \log(n), 2] = P$$

$$PCP_{1,0}[\log(n), 42, 2] = P$$

$$PCP_{1,0}[poly(n), 42, 2] = BPP$$

$$PCP_{1,1-1/m}[\log(n), 3, 2] = NP$$

$$PCP_{1,1/2}[poly\log(n), poly(\log n), 2] = NP \quad \text{(Lecture 10)}$$

PCP Theorem (version 2)

$$PCP_{1,1/2}[O(\log n), O(1)] \supseteq NP$$

Why equivalent?

Version 1 $\Longrightarrow$ Version 2

Take your favorite $L \in NP$

$\exists$ reduction R (poly-time) mapping L to Max 3-SAT s.t.

if $\varphi \in L$ then $\mathrm{Val}(R(\varphi)) = 1$

$\varphi \notin L$ then $\mathrm{Val}(R(\varphi)) \leq 1-\delta$

PCP verifier for L:

~~1~~ Expects as proof π an optimal assignment to $R(\varphi)$.

1. Pick t random clauses $C_1, \ldots, C_t$ of $R(\varphi)$.
2. Accept if all the clauses are satisfied

Parameters?

Randomness $r = t \cdot \log(\#\text{clauses}(R(\varphi)))$

$= t \cdot \log(n)$

Queries $= 3 \cdot t$

Completeness $= 1$ (why?)

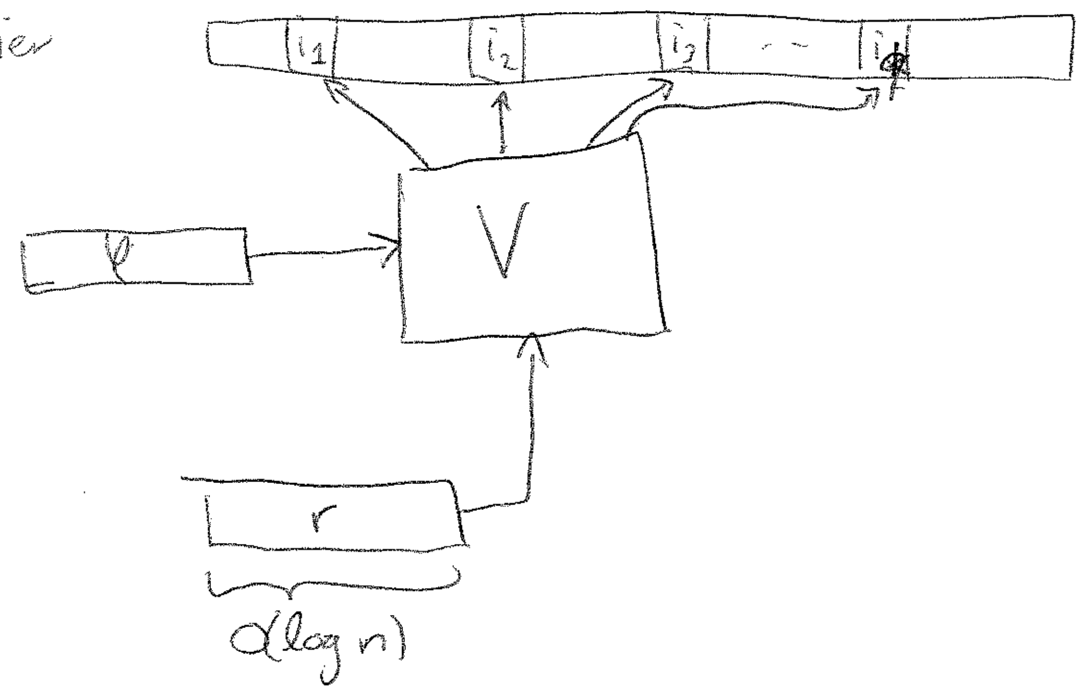
Soundness $= (1-\delta)^t$ (why?)

Pick $t = 1/\log\frac{1}{1-\delta} \simeq 1/\delta$ so that $(1-\delta)^t \leq 1/2$

$\longrightarrow L \in \mathrm{PCP}$ [illegible]

Version 2 $\Rightarrow$ Version 1

Picture of verifier



Observation The check that V does on π is completely determined by φ and r

φ is a fixed instance, there are $2^r = 2^{O(\log n)} = poly(n)$ many possibilities for r Write ~~these~~ these down:

r	verifier checks
00000...00	$C_1(\pi_2, \pi_7, \pi_9, \pi_{17})$
00000...01	$C_2(\cdot \; \cdot \; \cdot \; \cdot)$
... 10	$C_3(\cdot \; \cdot \; \cdot)$
⋮	
11111111	$C_{2^r}(\cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot)$

This is a Max q-CSP$_2$ instance φ' over the variables $\pi_1, \pi_2, \ldots, \pi_{|\pi|}$.

Observation

$$\max_{\pi} \Pr[V \text{ accepts } \pi] = \mathrm{val}(\varphi')$$

So PCP Thm version 2 says that there is a q s.t. Max q-CSP$_2$ NP-hard to distinguish between $\mathrm{val}(\varphi') = 1$ and $\mathrm{val}(\varphi') \leq 1/2$

Now reduce from Max k-CSP$_2$ to MAX 3-SAT to get version 1.

PCP Thm Proof Sketch

Lemma 22.4 (Main)

$\exists\, \varepsilon_0 > 0,\ C > 0$, polytime reduction R from Max 2-CSP_3 to Max 2-CSP_3

s.t. $|R(\varphi)| \le C \cdot |\varphi|$

- If $val(\varphi) = 1$ then $val(R(\varphi)) = 1$
- If $val(\varphi) \le 1 - \varepsilon$ for $\varepsilon < \varepsilon_0$ then

  $val(R(\varphi)) \le 1 - 2\varepsilon$

Proof of PCP Thm from Lemma 22.4

Given instance φ of Max 3-Coloring (i.e. of Max 3-CSP_{Σ}) with n vars, m clauses, we know it is NP-hard to distinguish between

$$val(\varphi) = 1 \quad \text{and} \quad val(\varphi) \le 1 - 1/m$$

(Cook's Thm!)

Apply R from Lemma 22.4 t times. Get gap

$val(R^{(t)}(\varphi)) = 1$ vs. $val(R^{(t)}(\varphi)) \le 1 - 2^t/m$,

assuming $2^t/m \le \varepsilon_0$

Set $t \approx \log_2(m/\varepsilon_0)$ so that $2^t/m \in [\varepsilon_0/2, \varepsilon_0]$

Done?

what is size?

$|R^{(t)}(\varphi)| \leq C \cdot |R^{(t-1)}(\varphi)| \leq \dots \leq C^t \cdot |\varphi|$

$= \text{poly}(m) |\varphi|$ $\square$

<u>Def</u>: CL-reduction ("complete linear-blowup reduction")

R from Max q-CSP_W to Max q'-$CSP_{W'}$:

* Perfect completeness ($\text{val}=1$ to $\text{val}=1$)
* poly-time
* Constant factor size blowup
  ($\exists$ universal const. C s.t. $|R(\varphi)| \leq C \cdot |\varphi|$

<u>Lemma 22.5</u>

$\forall \ell$ $\exists W, \varepsilon_0 > 0$ and CL-reduction R_ℓ^+ from Max 2-CSP_3 to Max 2-CSP_W s.t. if $\text{val}(\varphi) \leq 1-\varepsilon$ for $\varepsilon \leq \varepsilon_0$ then

$\text{val}(R_\ell^+(\varphi)) \leq 1 - \ell \cdot \varepsilon$