Lecture 6

KTH Stockholm

February 28, 2014

Greatest Common Divisors

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- **The** GCD is the **positive** GCD.
- We denote the GCD of m and n by $\gcd(m, n)$.

Properties

- $\gcd(m, n) = \gcd(n, m)$
- $\gcd(m, n) = \gcd(m \pm n, n)$
- $\gcd(m, n) = \gcd(m \bmod n, n)$
- $\gcd(m, n) = 2\gcd(m/2, n/2)$ if m and n are even.
- $\gcd(m, n) = \gcd(m/2, n)$ if m is even and n is odd.

Euclidean Algorithm

EUCLIDEAN(m, n)
(1) **while** $n \neq 0$
(2) $t \leftarrow n$
(3) $n \leftarrow m \bmod n$
(4) $m \leftarrow t$
(5) **return** m

Steins Algorithm (Binary GCD Algorithm)

```
Stein(m, n)
    if m = 0 or n = 0 then return 0
    s ← 0
    while m and n are even
        m ← m/2, n ← n/2, s ← s + 1
    while n is even
        n ← n/2
    while m ≠ 0
        while m is even
            m ← m/2
        if m < n
            Swap(m, n)
        m ← m − n
        m ← m/2
    return 2^s m
```

Bezout's Lemma

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Lemma. There exists integers a and b such that

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Proof. Let $d > \gcd(m, n)$ be the smallest positive integer on the form $d = am + bn$. Write $m = cd + r$ with $0 < r < d$. Then

$$d > r = m - cd = m - c(am + bn) = (1 - ca)m + (-cb)n \ ,$$

a contradiction! Thus, $r = 0$ and $d \mid m$. Similarly, $d \mid n$.

Extended Euclidean Algorithm (Recursive Version)

ExtendedEuclidean(m, n)
(1) **if** $m \bmod n = 0$
(2) **return** $(0, 1)$
(3) **else**
(4) $(x, y) \leftarrow$ ExtendedEuclidean($n, m \bmod n$)
(5) **return** $(y, x - y\lfloor m/n \rfloor)$

If $(x, y) \leftarrow$ ExtendedEuclidean(m, n) then $\gcd(m, n) = xm + yn$.

Coprimality (Relative Primality)

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Excercise: Why is this so?

Chinese Remainder Theorem (CRT)

Theorem. (Sun Tzu 400 AC) Let $n_1, \ldots, n_k$ be positive pairwise coprime integers and let $a_1, \ldots, a_k$ be integers. Then the equation system

$$\begin{aligned} x &= a_1 \bmod n_1 \\ x &= a_2 \bmod n_2 \\ x &= a_3 \bmod n_3 \\ &\vdots \\ x &= a_k \bmod n_k \end{aligned}$$

has a unique solution in $\{0, \ldots, \prod_i n_i - 1\}$.

Constructive Proof of CRT

1. Set $N = n_1 n_2 \cdot \ldots \cdot n_k$.
2. Find r_i and s_i such that $r_i n_i + s_i \frac{N}{n_i} = 1$ (Bezout).
3. Note that

$$s_i \frac{N}{n_i} = 1 - r_i n_i = \begin{cases} 1 \pmod{n_i} & \\ 0 \pmod{n_j} & \text{if } j \neq i \end{cases}$$

4. The solution to the equation system becomes:

$$x = \sum_{i=1}^{k} \left(s_i \frac{N}{n_i} \right) \cdot a_i$$

The Multiplicative Group

The set $\mathbb{Z}_n^* = \{0 \leq a < n \;:\; \gcd(a,n) = 1\}$ forms a group, since:

- **Closure.** It is closed under multiplication modulo n.
- **Associativity.** For $x, y, z \in \mathbb{Z}_n^*$:

$$(xy)z = x(yz) \bmod n \ .$$

- **Identity.** For every $x \in \mathbb{Z}_n^*$:

$$1 \cdot x = x \cdot 1 = x \ .$$

- **Inverse.** For every $a \in \mathbb{Z}_n^*$ exists $b \in \mathbb{Z}_n^*$ such that:

$$ab = 1 \bmod n \ .$$

Lagrange's Theorem

Theorem. If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.

Proof.

1. Define $aH = \{ah : h \in H\}$. This gives an equivalence relation $x \approx y \Leftrightarrow x = yh \wedge h \in H$ on G.
2. The map $\phi_{a,b} : aH \rightarrow bH$, defined by $\phi_{a,b}(x) = ba^{-1}x$ is a bijection, so $|aH| = |bH|$ for $a, b \in G$.

Euler's Phi-Function (Totient Function)

Definition. Euler's Phi-function $\phi(n)$ counts the number of integers $0 < a < n$ relatively prime to n.

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- In general: $\phi\left(\prod_i p_i^{k_i}\right) = \prod_i \left(p_i^k - p_i^{k-1}\right)$.

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  Excercise: How does this follow from CRT?

Fermat's and Euler's Theorems

Theorem. (Fermat) If $b \in \mathbb{Z}_p^*$ and p is prime, then $b^{p-1} = 1 \bmod p$.

Theorem. (Euler) If $b \in \mathbb{Z}_n^*$, then $b^{\phi(n)} = 1 \bmod n$.

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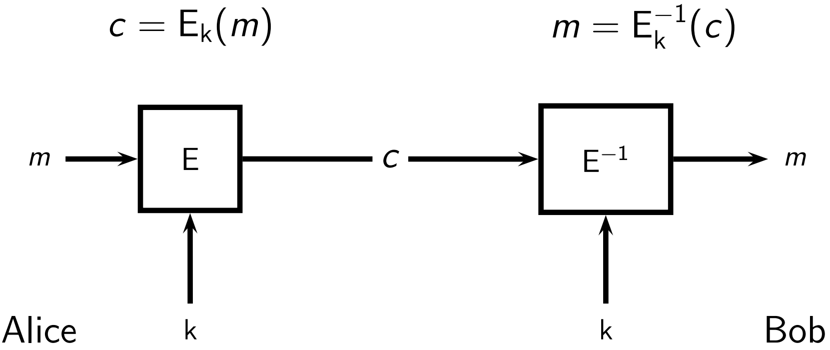
Proof. Note that $|\mathbb{Z}_n^*| = \phi(n)$. b generates a subgroup $\langle b \rangle$ of $\mathbb{Z}_n^*$, so $|\langle b \rangle|$ divides $\phi(n)$ and $b^{\phi(n)} = 1 \bmod n$.

Multiplicative Group of a Prime Order Field

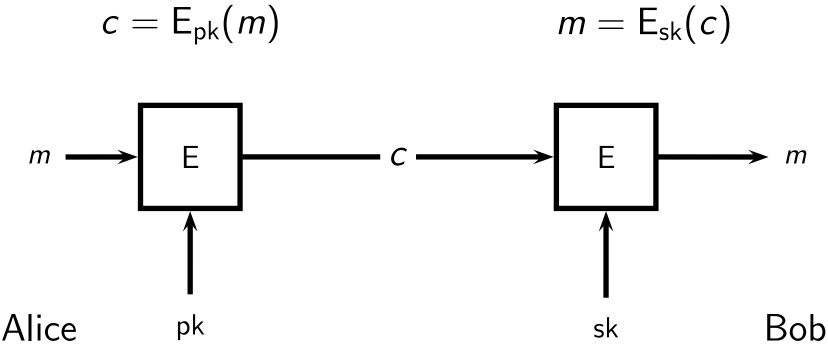
Definition. A group G is called **cyclic** if there exists an element g such that each element in G is on the form g^x for some integer x.

Theorem. If p is prime, then $\mathbb{Z}_p^*$ is cyclic.

Cipher (Symmetric Cryptosystem)



Public-Key Cryptosystem



History of Public-Key Cryptography

Public-key cryptography was discovered:

- By Ellis, Cocks, and Williamson at the Government Communications Headquarters (GCHQ) in the UK in the early 1970s (not public until 1997).
- Independently by Merkle in 1974 (Merkle's puzzles).
- Independently in its discrete-logarithm based form by Diffie and Hellman in 1977, and instantiated in 1978 (key-exchange).
- Independently in its factoring-based form by Rivest, Shamir and Adleman in 1977.

Public-Key Cryptography

Definition. A public-key cryptosystem is a tuple $(\mathrm{Gen}, \mathrm{E}, \mathrm{E})$ where,

- Gen is a **probabilistic key generation algorithm** that outputs key pairs $(\mathrm{pk}, \mathrm{sk})$,
- E is a (possibly probabilistic) **encryption algorithm** that given a public key pk and a message m in the plaintext space $\mathcal{M}_{\mathrm{pk}}$ outputs a ciphertext c, and
- E is a **decryption algorithm** that given a secret key sk and a ciphertext c outputs a plaintext m,

such that $\mathrm{E}_{\mathrm{sk}}(\mathrm{E}_{\mathrm{pk}}(m)) = m$ for every $(\mathrm{pk}, \mathrm{sk})$ and $m \in \mathcal{M}_{\mathrm{pk}}$.

The RSA Cryptosystem (1/2)

Key Generation.

- Choose $n/2$-bit primes p and q randomly and define $N = pq$.
- Choose e in $\mathbb{Z}^*_{\phi(N)}$ and compute $d = e^{-1} \bmod \phi(N)$.
- Output the key pair $((N, e), (p, q, d))$, where (N, e) is the public key and (p, q, d) is the secret key.

The RSA Cryptosystem (2/2)

Encryption. Encrypt a plaintext $m \in \mathbb{Z}_N^*$ by computing

$$c = m^e \bmod N \enspace .$$

Decryption. Decrypt a ciphertext c by computing

$$m = c^d \bmod N \enspace .$$

Why Does It Work?

$$(m^e \bmod N)^d \bmod N = m^{ed} \bmod N$$

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(m^e \bmod N)^d \bmod N &= m^{ed} \bmod N \\
&= m^{1+t\phi(N)} \bmod N
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(m^e \bmod N)^d \bmod N &= m^{ed} \bmod N \\
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&= m \bmod N
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Implementing RSA

- Modular arithmetic.
- Primality test.

Modular Arithmetic (1/2)

Basic operations on $O(n)$-bit integers using "school book" implementations.

Operation	Running time
Addition	$O(n)$
Subtraction	$O(n)$
Multiplication	$O(n^2)$
Modular reduction	$O(n^2)$

What about modular exponentiation?

Modular Arithmetic (2/2)

Square-and-Multiply.

SquareAndMultiply(x, e, N)

$z \leftarrow 1$
$i =$index of most significant bit
while $i \geq 0$
    do
        $z \leftarrow z \cdot z \bmod N$
        if $e_i = 1$
            then $z \leftarrow z \cdot x \bmod N$
        $i \leftarrow i - 1$
return z

Prime Number Theorem

The primes are relatively dense.

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Theorem. Let $\pi(m)$ denote the number of primes $0 < p \leq m$. Then

$$\lim_{m \to \infty} \frac{\pi(m)}{\frac{m}{\ln m}} = 1 \ .$$

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To generate a random prime, we repeatedly pick a random integer m and check if it is prime. It should be prime with probability $1/\ln m$.

Legendre Symbol (1/2)

Definition. Given an odd integer $b \geq 3$, an integer a is called a **quadratic residue** modulo b if there exists an integer x such that $a = x^2 \bmod b$.

Definition. The **Legendre Symbol** of an integer a modulo an **odd prime** p is defined by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } a = 0 \\ 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is a quadratic non-residue modulo } p \end{cases} .$$

Legendre Symbol (2/2)

Theorem. If p is an odd prime, then

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Proof.

- If $a = y^2 \bmod p$, then $a^{(p-1)/2} = y^{p-1} = 1 \bmod p$.

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$$\left(\frac{a}{p}\right) = a^{(p-1)/2} \bmod p \ .$$

Proof.

- If $a = y^2 \bmod p$, then $a^{(p-1)/2} = y^{p-1} = 1 \bmod p$.
- If $a^{(p-1)/2} = 1 \bmod p$ and b generates $\mathbb{Z}_p^*$, then $a^{(p-1)/2} = b^{x(p-1)/2} = 1 \bmod p$ for some x. Since b is a generator, $(p-1) \mid x(p-1)/2$ and x must be even.

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- If $a^{(p-1)/2} = 1 \bmod p$ and b generates $\mathbb{Z}_p^*$, then $a^{(p-1)/2} = b^{x(p-1)/2} = 1 \bmod p$ for some x. Since b is a generator, $(p-1) \mid x(p-1)/2$ and x must be even.
- If a is a non-residue, then $a^{(p-1)/2} \neq 1 \bmod p$, but $\left(a^{(p-1)/2}\right)^2 = 1 \bmod p$, so $a^{(p-1)/2} = -1 \bmod p$.

Jacobi Symbol

Definition. The **Jacobi Symbol** of an integer a modulo an odd integer $b = \prod_i p_i^{e_i}$, with p_i prime, is defined by

$$\left(\frac{a}{b}\right) = \prod_i \left(\frac{a}{p_i}\right)^{e_i} \quad .$$

Note that we can have $\left(\frac{a}{b}\right) = 1$ even when a is a non-residue modulo b.

Properties of the Jacobi Symbol

Basic Properties.

$$\left(\frac{a}{b}\right) = \left(\frac{a \bmod b}{b}\right)$$
$$\left(\frac{ac}{b}\right) = \left(\frac{a}{b}\right)\left(\frac{c}{b}\right) \quad .$$

Law of Quadratic Reciprocity. If a and b are odd integers, then

$$\left(\frac{a}{b}\right) = (-1)^{\frac{(a-1)(b-1)}{4}}\left(\frac{b}{a}\right) \quad .$$

Supplementary Laws. If b is an odd integer, then

$$\left(\frac{-1}{b}\right) = (-1)^{\frac{b-1}{2}} \quad \text{and} \quad \left(\frac{2}{b}\right) = (-1)^{\frac{b^2-1}{8}} \quad .$$

Computing the Jacobi Symbol (1/2)

The following assumes that $a \geq 0$ and that $b \geq 3$ is odd.

$\textsc{Jacobi}(a, b)$

if $a < 2$
    return a
$s \leftarrow 1$
while a is even
    $s \leftarrow s \cdot (-1)^{\frac{1}{8}(b^2-1)}$
    $a \leftarrow a/2$
if $a < b$
    $\textsc{Swap}(a,b)$
    $s \leftarrow s \cdot (-1)^{\frac{1}{4}(a-1)(b-1)}$
return $s \cdot \textsc{Jacobi}(a \bmod b, b)$

Solovay-Strassen Primality Test (1/2)

The following assumes that $n \geq 3$.

$\textsc{SolovayStrassen}(n, r)$
(1) **for** $i = 1$ **to** r
(2) Choose $0 < a < n$ randomly.
(3) **if** $\left(\frac{a}{n}\right) = 0$ or $\left(\frac{a}{n}\right) \neq a^{(n-1)/2} \bmod n$
(4) **return** *composite*
(5) **return** *probably prime*

Solovay-Strassen Primality Test (2/2)

Analysis.

- If m is prime, then $0 \neq \left(\frac{a}{m}\right) = a^{(m-1)/2} \bmod m$ for all $0 < a < m$, so we never claim that a prime is composite.

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- If $\left(\frac{a}{m}\right) = 0$, then $\left(\frac{a}{p}\right) = 0$ for some prime factor p of m. Thus, $p \mid a$ and m is composite, so we never wrongly return from within the loop.

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- At most half of all elements a in $\mathbb{Z}_m^*$ have the property that

$$\left(\frac{a}{m}\right) = a^{(m-1)/2} \bmod m \ .$$