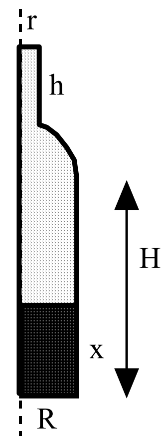
Design of Bourgogne Organ

1 Background



The task is to determine the frequency of the sound obtained by blowing across the mouth of a partially filled standard Bourgogne bottle, and to select the liquid levels to make a properly tuned bottle organ.
The rotationally symmetric geometry is defined by the bottom radius *R*, also the radius of the spherical shoulder segment, the height *H*, the neck radius *r* and length *h*, the liquid level is *x*.

2 Helmholtz Resonator Model

In the Swedish High-school national physics contest 1994 one learns that the contraption is a Helmholtz resonator, where the air in the bottleneck acts as a rigid body, and the (large) volume in the bottle itself acts as a spring. The test suggests the use of

$$pV = \text{const.},$$

i.e. assuming that the process is isothermal. This is erroneous and gives about 20% error, see below, so one should use instead, with γ = ratio of specific heats for air = 1.4,

$$\mathrm{d}p/\mathrm{d}V = -\gamma\, p/V$$

i.e.

$$pV^{\gamma} = \text{const.}$$

One can deduce from this the lowest resonance frequency.

3 Analytical and Preparation Tasks

Q1. Do that. You may assume that $H = 3R$, $R = 3r$, and volume = 3/4 litres, that the temperature is 20^{o} C, and standard air pressure of 10^5 Pa, at which conditions the density of air is 1.273 kg/m^3. Plot the frequency as function of *x*, $0 \le x \le H$.
Hint: The air plug has mass *m* (= …) and is accelerated by the pressure difference acting on its cross section area *A*. If it has velocity *v*, positive outward, then the volume rate of change of the air in the large volume *V* is

$$\mathrm{d}V/\mathrm{d}t = Av$$

But $\mathrm{d}p/\mathrm{d}t = \mathrm{d}p/\mathrm{d}V\ \mathrm{d}V/\mathrm{d}t$ so we now have two first order equations for *p* and *v* from which the resonance frequency can be obtained (by linearization).

4 COMSOL Calculations

4.1. Now let us do this by eigenvalue computation in COMSOL. It follows from thermodynamics that air pressure deviations *p* from ambient pressure *P* are described by the wave equation,

$$p_{tt} = c^2\,\Delta p$$
$$\partial p/\partial n = 0 \text{ at solid boundaries}$$
$$p = 0 \text{ at the mouth}$$

c is the adiabatic speed of sound, 340 m/s at the conditions above.
Q2 a) Plot the frequency as function of *x*, $0 \le x \le H$, in the same diagram as the curve in 1.
Q3 b) Comment on the differences between the Helmholtz resonator results and the – probably more accurate – COMSOL results.
READ QUESTION 3 BEFORE DOING THE CALCULATIONS!

4.2. The variation of *x* is conceptually most easily treated by using the "**`parametrized geometry`**" Module. An option is to include the pressure waves in the liquid so the liquid level becomes a parameter which can be changed without changing the geometry. Here is how:

4.2.1. **Liquid**

The liquid is almost incompressible and the correct model is

$$(E)^{-1}p_{tt} = \mathrm{div}(1/\rho \text{ grad } p)$$

where E is the elastic modulus (N/m^2) and ρ the density (1000 kg/m^3). It is known that the sound speed in water is about 1500 m/s.

Q4) Deduce the value of E from this. Hint: With constant ρ and E we obtain $p_{tt} = E/\rho\ \Delta p$

4.2.2. **Gas**

The equation for the gas is

$$(c^2\rho)^{-1}p_{tt} = \mathrm{div}(1/\rho \text{ grad } p)$$

4.2.3. **Interface conditions**

For the whole model, p and $\partial p/\partial n/\rho$ must be continuous across the interface. This comes automatically with the FE formulation.

Define ρ as

$$\rho(r,z) = Hs(z-x)\rho_{Gas} + (1-Hs(z-x))\rho_{Liq}$$

where $Hs(s)$ is a smoothed Heaviside function jumping from 0 to 1 at $s = 0$. A similar definition is made of the coefficient of p_{tt} (called e_a in COMSOL),

$$e_a(r,z) = Hs(z-x)\frac{1}{\rho_{Gas}c^2} + (1-Hs(z-x))\frac{1}{E_{Liq}}$$

4.3 *Hs* jumps over a distance d, a parameter to the `FLH2S` function built into COMSOL. One suspects that d must be O(element size) so the jump occurs over a few elements lest the quadratures become inaccurate.

Q5 a) Experiment with small d. Look at the pressure gradient near the interface. If it looks spiky d is too small.

Q6 b) Run the computations. Compare a few with results obtained by changing the geometry.

Q7 5. So armed with the tables, select the eight liquid levels.

5 COMSOL implementation

5.1 Mode

5.2 Geometry

5.3 Constants

5.4 Expressions

5.5 Solver Settings

5.6 Convergence Study

Appendix

Adiabatic relation between pressure *p* and Volume *V*

Assume that the process is adiabatic, i.e. no heat exchange with the outside. Then, for the volume in the bottle

$$0 = \text{added heat} = dQ = dE + p\, dV$$

where

$$p = m/V\, R\, T,\ R = \text{universal gas constant}$$

Assuming the gas to be calorically perfect, its internal energy per unit mass *e* depends only on temperature:

$$E = me = m\, c_v\, T = p\, c_v\, V/R,$$

so

$$0 = c_v/R\, d(pV) + p\, dV = (c_v/R + 1)\, p\, dV + c_v/R\, V\, dp;$$

$$dp/dV|_{dQ=0} = -\,(c_v + R)/c_v\, p/V = -\,\gamma\, p/V$$

where γ is 7/5 = 1.4, a value which can be deduced from the kinetic theory of gases for two-atomic gases such as air. The isothermal process has

$$dp/dV|_{dT=0} = -\, p/V$$

so the error is 40% which translates into an error of 20% in wave speed. Use the correct relation in problem 1.