Finita Elementmetoden + Projektbeskrivningar

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Plan och Syfte

- Admin: tenta
- Randvillkor
- Andraderivata-matris
- Projektbeskrivningar
- FEM/andra metoder: fördelar nackdelar
- Modul 6 (del av projekt)

Randvärdesproblem

$$\begin{aligned} \text{Ekvation:} \quad & -u(x)'' = f(x), x \in [a, b] \\ \text{Randvillkor typ 1:} \quad & u(a) = 0, \quad u(b) = 0 \\ \text{Randvillkor typ 2:} \quad & u'(a) = 0, \quad u'(b) = 0 \end{aligned}$$

[Demo]

Vi kan skriva ett generellt randvillor som:

$$\text{Randvillkor generellt:} \quad u'(a) = \kappa u(a), \quad u'(b) = \kappa u(b)$$

$\kappa = 0$ blir typ 2, $\kappa \to \infty$ blir typ 1

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$$\int_a^b(u'\phi_j' - f\phi_j)dx + u'(b)\phi_j(b) - u'(a)\phi_j(a) = 0 \quad \text{[PI]}$$

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Använd generellt randvillkor $u'(a) = \kappa u(a)$:

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Randvillkor i Python/DOLFIN

```
from dolfin import *

# Coefficient for boundary condition
class Kappa(Expression):
    def eval(self, values, x):
        if(x[0] < 0.5):
            values[0] = 0.0
        else:
            values[0] = 1.0e6

# Mesh and basis functions (function space)
mesh = UnitInterval(8)
V = FunctionSpace(mesh, "CG", 1)

# FEM formulation
u = TrialFunction(V)
v = TestFunction(V)
f = Expression("1.0")
kappa = Kappa(V)
# Equation: -u'' = f
a = (inner(grad(u), grad(v)))*dx + kappa*u*v*ds
L = (f*v)*dx

# Build linear system and solve for coefficients uh in linear combination
problem = VariationalProblem(a, L)
uh = problem.solve()
plot(uh)
```

Andraderivata - matris

Använd definition av u som linjärkombination av basfunktioner:

$$u = \sum_{i=1}^{J} u_i \phi_i(x)$$

$$\int_a^b u' \phi_j' = \sum_{i=1}^{J} \int_a^b u_i \phi_i' \phi_j' dx \quad j = 1, 2, ..., J$$

Alltså, för matriselement $A_i j$:

$$A_{ij} = \int_a^b \phi_j' \phi_i' dx,$$

Andraderivata - matris

```
from dolfin import *

# Create mesh and define function space
mesh = UnitInterval(2)
V = FunctionSpace(mesh, ''CG'', 1)

# Define FEM formulation
u = TrialFunction(V)
v = TestFunction(V)
a = (inner(grad(u), grad(v)))*dx

A = assemble(a)
print A.array()

...

Matrix of size 3 x 3 has 7 nonzero entries.
[[ 2. -2.  0.]
 [-2.  4. -2.]
 [ 0. -2.  2.]]
```

Python help

```
>>> help(Mesh)
>>> mesh = UnitInterval(2)
>>> c = cells(mesh)
```

Projektbeskrivningar

The 3 parts of the reports are:

A Determine relevant data and use an ODE model to simulate your problem, analyze your results in terms of the underlying problem, and comment on possible sources of errors.

B Determine relevant data and formulate a PDE model for your problem.

C Use your PDE model from B to simulate your problem, analyze the results in terms of the underlying problem, and comment on possible sources of errors.

Projektbeskrivningar: Konvektion-diffusion-reaktion

The convection-diffusion-reaction of the concentration of one species u_1, which is consumed by another species of concentration u_2, can be described by the following ODE and PDE models:

ODE:

$$\dot{u}_1 = -\alpha_1 u_1 u_2 \tag{1}$$

$$\dot{u}_2 = \alpha_2 u_1 u_2 - \alpha_3 u_2 \tag{2}$$

PDE:

$$\dot{u} + \beta \cdot \nabla u - \epsilon \Delta u = f(u) \tag{3}$$

$$\left[\dot{u} + \beta u' - \epsilon u'' = f(u)\right] \tag{4}$$

Projektbeskrivningar: Vågutbredning

Wave phenomena can be modeled by a mass-spring model in an ODE, or using a PDE wave equation:

ODE:

$$\dot{x}^i = v^i \quad \dot{v}^i = \frac{F^i}{m^i} \tag{5}$$

$$F^i = \sum_{j=0}^{N} F^{ij} \quad F^{ij} = E(r^{ij} - L^{ij})e^{ij} \tag{6}$$

PDE:

$$\dot{u}_1 = u_2 \tag{7}$$

$$\dot{u}_2 - c^2 \Delta u_1 = 0 \tag{8}$$

$$\left[\dot{u}_2 - c^2 u_1'' = 0\right] \tag{9}$$

Elasticity can be modeled by a mass-spring model in an ODE, or using a PDE model:

ODE:

$$\dot{x}^i = v^i \quad \dot{v}^i = \frac{F^i}{m^i} \tag{10}$$

$$F^i = \sum_{j=0}^{N} F^{ij} \quad F^{ij} = E(r^{ij} - L^{ij})e^{ij} \tag{11}$$

PDE:

$$\dot{u}_1 = u_2 \tag{12}$$

$$\dot{u}_2 - \nabla \cdot \sigma = 0 \tag{13}$$

$$\dot{\sigma} = \frac{1}{2}E(\nabla u_2 + \nabla u_2^T) \tag{14}$$