(b) If $a_{11}a_{22} - a_{12}a_{21} \neq 0$, show that

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

15. Let D be an $n \times n$ diagonal matrix. Show that $\det(D) = d_{11}d_{22}d_{33}\cdots d_{nn}$.

16. Let α be a real number and let

$$A = \begin{bmatrix} \alpha & 4 \\ 1 & \alpha \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 2 & \alpha & 0 \\ -3 & -1 & 5 \\ 1 & 3 & \alpha \end{bmatrix}.$$

(a) For what value(s) of α is A singular?
(b) For what value(s) of α is B singular?

3.1 GAUSSIAN ELIMINATION

In this chapter we study techniques for the solution of systems of linear algebraic equations. The most general system of n linear equations in n unknowns can be written as

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & b_3 \\ \vdots & & \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & + & a_{nn}x_n & = & b_n. \end{array}$$

The a_{ij} and the b_i are known constants, and the x_i are the variables. This system can be expressed very compactly in matrix notation as $A\mathbf{x} = \mathbf{b}$, where A is the $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ & \vdots & & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

and $\mathbf{x}$ and $\mathbf{b}$ are the n-dimensional column vectors $\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_n \end{bmatrix}^T$ and $\begin{bmatrix} b_1 & b_2 & b_3 & \cdots & b_n \end{bmatrix}^T$, respectively. A is called the *coefficient matrix*, $\mathbf{x}$ the *solution vector* and $\mathbf{b}$ the *right-hand side vector* for the system.

We will focus on the solution technique known as Gaussian elimination with back substitution. After a review of the basic algorithm, several examples will be presented to demonstrate the technique. Finally, a detailed account of the number of operations required to compute the solution will be given, and a comparison with other possible solution strategies will be made.