F8

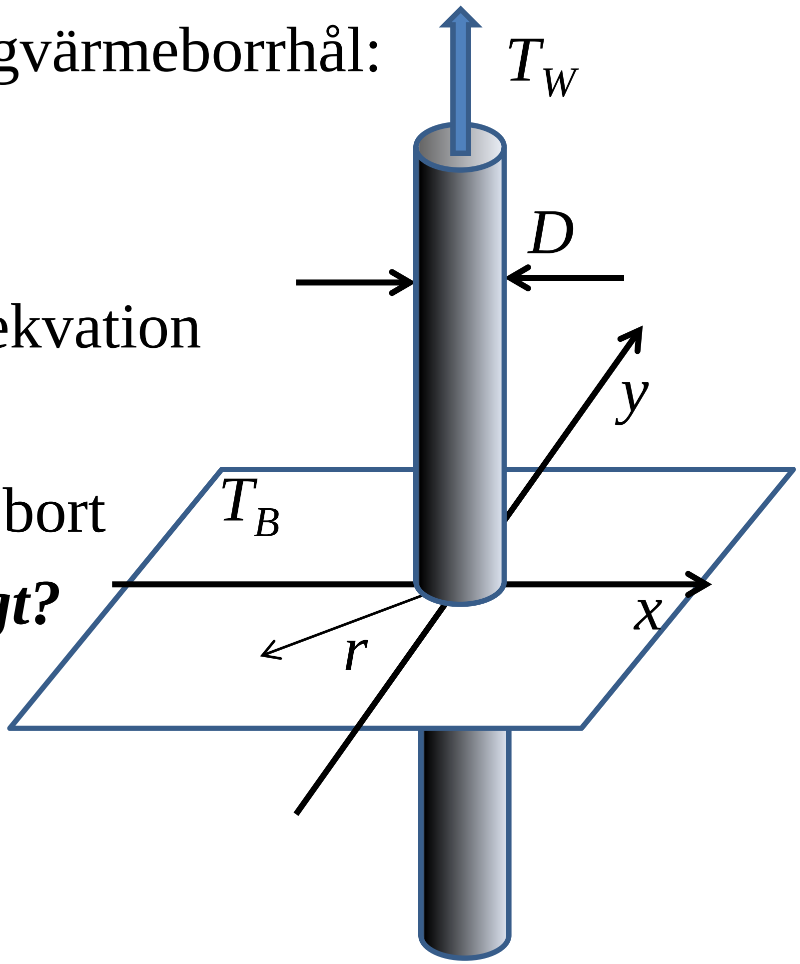
- Randvärdesproblem NAM 8 , exempel
- Differensmetod
- Inskjutning
- Formuleringar

Randvärdesproblem, I

Exempel: Hur stor effekt per m borrhål?

Temperaturfördelning runt lååångt bergvärmeborrhål:

Fourier's lag och axialsymmetri:



$$\rho C \frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(rk\frac{\partial T}{\partial r}\right)$$

Differentialekvation

$$T(\infty, t) = T_B$$

Temp. långt bort

2D! rimligt?

$$k\frac{\partial T}{\partial r}\bigg|_{r=D/2} = -\alpha(T_W - T)$$

Värmeövergång, vattnet

$$P = \pi D k \frac{\partial T}{\partial r}\bigg|_{r=D/2}$$

Effekt

Randvärdesproblem, differensmetod, I

Stationärt: d/dt = 0

Linjärt problem ”AT=b”

lokalt fel O(h²)

$$\frac{1}{r}\frac{d}{dr}\left(rk\frac{dT}{dr}\right)=0:\frac{dT}{dr}+r\frac{d^2T}{dr^2}=0$$

$$T(R)=T_B$$

$$\left.\frac{dT}{dr}\right|_{r=D/2}=-\frac{\alpha}{k}(T_W-T)$$

$$i=1,2,\ldots,N-1$$

$$\frac{dT}{dr}(r=r_i)\to\frac{T_{i+1}-T_{i-1}}{2h}$$

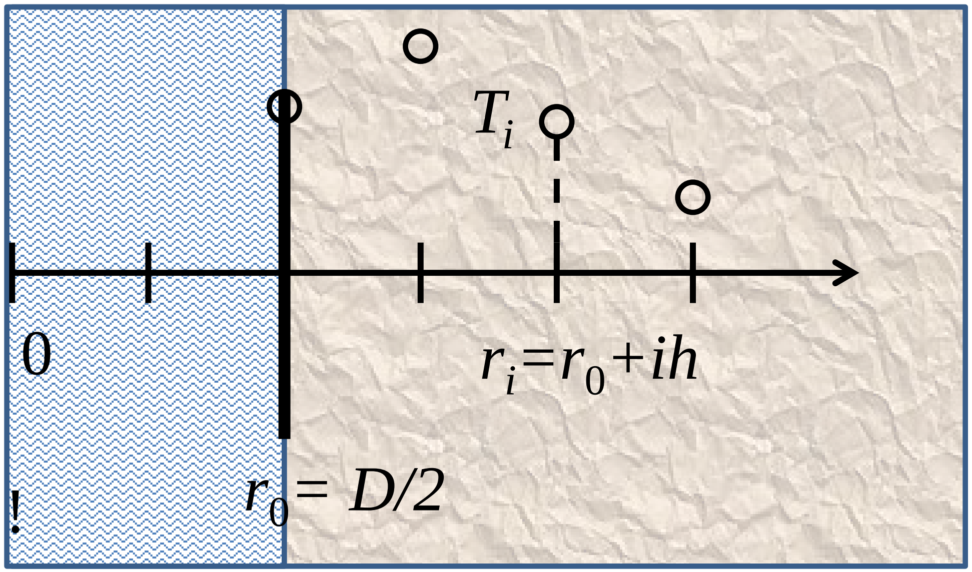
$$\frac{d^2T}{dr^2}(r=r_i)\to\frac{T_{i+1}-2T_i+T_{i-1}}{h^2}$$

$$T_N=T_B$$

$$\left.\frac{dT}{dr}\right|_{r=D/2}=-\frac{\alpha}{k}(T_W-T)\to$$

$$\frac{T_1-T_0}{h}+\frac{\alpha}{k}(T_W-T_0)=0$$

Fel O(h) !



Randvärdesproblem, differensmetod II

$$\frac{T_1 - T_0}{h} + \frac{\alpha}{k}(T_W - T_0) = 0:$$

$$-(\frac{\alpha}{k} + \frac{1}{h})T_0 + \frac{1}{h}T_1 = -\frac{\alpha}{k}T_W, i = 0$$

$$\frac{T_{i+1} - T_{i-1}}{2h} + r_i \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} = 0:$$

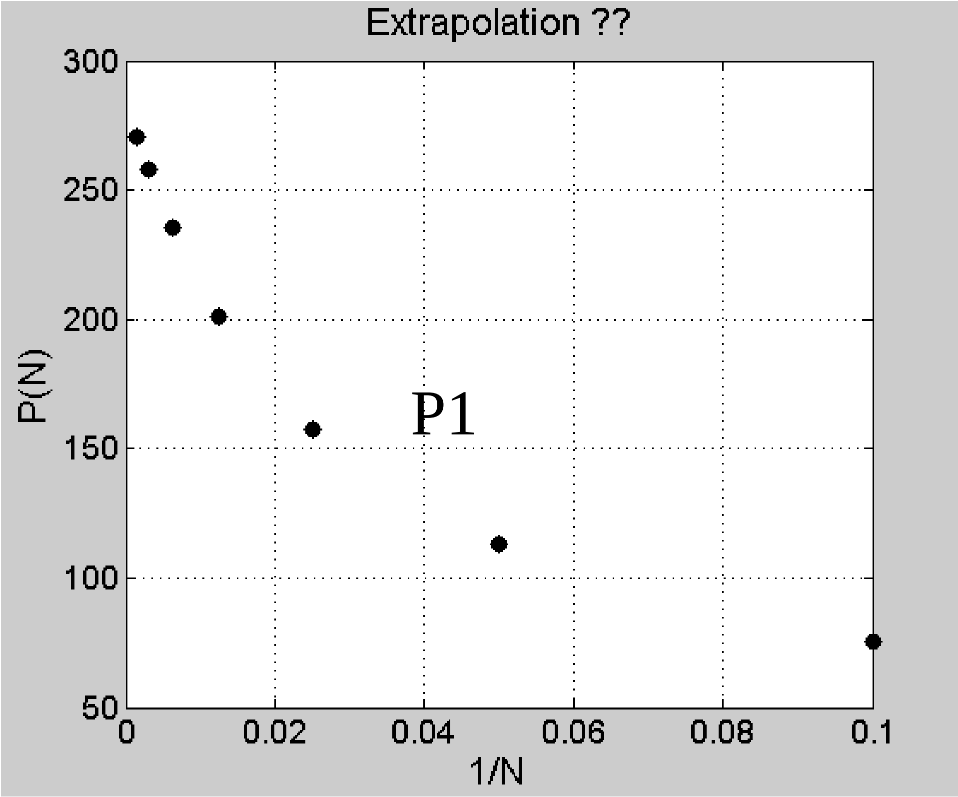
$$(-\frac{1}{2h} + \frac{r_i}{h^2})T_{i-1} + (-\frac{2r_i}{h^2})T_i + (\frac{1}{2h} + \frac{r_i}{h^2})T_{i+1} = 0,$$

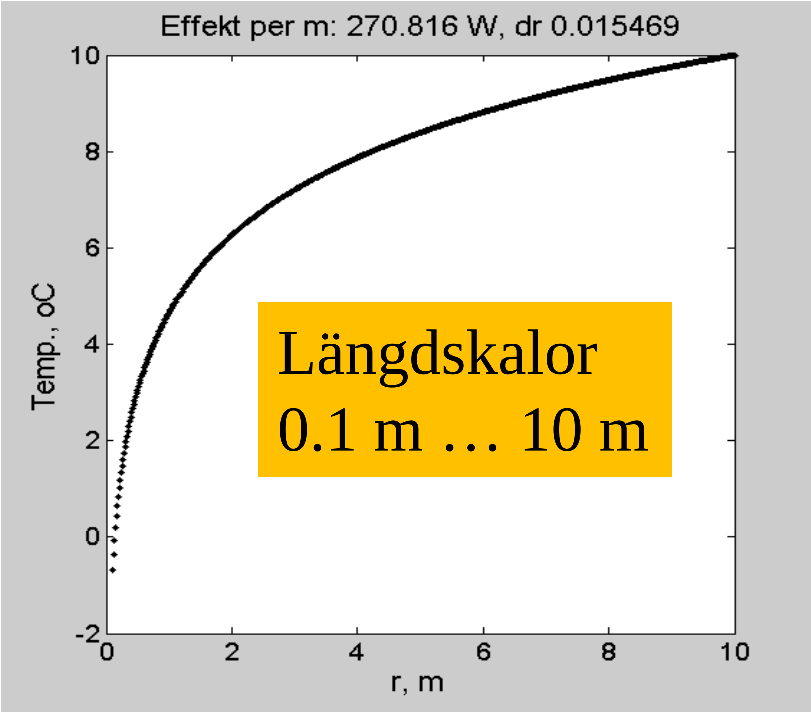
$$T_N = T_B, i = N$$

```
N = 8;      % T(1...N+1)
r = linspace(D/2,R,N+1);
h = r(2)-r(1); h2 = h*h;
A = zeros(N+1,N+1);
b = zeros(N+1,1);

A(1,1) = -(alfa/k+1/h);
A(1,2) = 1/h;
b(1)   = -alfa/k*TW;

for j = 2:N % k = i+1
  rj = r(j);
  A(j,j-1) = -1/(2*h)+rj/h2;
  A(j,  j) = -2*rj/h2;
  A(j,j+1) = 1/(2*h)+rj/h2;
end

A(N+1,N+1) = 1;
b(N+1) = TB;
```

Randvärdesproblem, differensmetod III

N	P1	P2
10	75.7	
20	113.1	141.4
40	157.8	190.8
80	201.2	234.0
160	235.3	262.6
320	257.8	277.0
640	270.8	282.7
1280	277.9	284.6





Fel $O(h)$ … oekonomiskt ?
Förbättra approx. av dT/dr
i randvillkoret (P2)

$$\frac{dT}{dr}(r_0) \to (-3/2 \cdot T_0 + 2 \cdot T_1 - 1/2 \cdot T_2)/h, \text{Fel}\, O(h^2)$$

Randvärdesproblem, differensmetod II

Lös ekv. syst. **res = AT - b = 0**
Jacobian-matrisen kan konstrueras med numerisk differentieríng som i Newtons metod

```
function res = tempbvpres(T)
global TW TB D k alfa R
n = length(T);
res = 0*T;
r = linspace(D/2,R,n)';
h = r(2)-r(1);
h2 = h*h;
% BVC at r = D/2 T(1)
res(1) = (T(2)-T(1))/h + …
alfa/k*(TW-T(1));
% the inner points
D2T = diff(diff(T));
res(2:n-1) = (T(3:n)-T(1:n-2))/(2*h) +
             r(2:n-1).*D2T/h2;
% BVC at r = R
res(n) = T(n)-TB;
%================================
N = 80;
b = -tempbvpres(zeros(N,1));
A = jacobian('tempbvpres',zeros(N,1),1e-4); T = A \ b;
```

$$res(1) = \frac{T_2 - T_1}{h} + \frac{\alpha}{k}(T_W - T_1)$$

$$res(i) = \frac{T_{i+1} - T_{i-1}}{2h} + \dots$$
$$+ r_i \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}, i = 2,\dots,n-1$$

$$res(n) = T_n - T_B$$

Randvärdesproblem, inskjutningsmetod I

$$\frac{dT}{dr} + r\frac{d^2T}{dr^2} = 0, T(R) = T_B, \left.\frac{dT}{dr}\right|_{r=D/2} = -\frac{\alpha}{k}(T_W - T)$$

$$\Rightarrow$$

$$\begin{cases} u_1' = u_2, \qquad u_1(R) = T_B \\ u_2' = -u_2/r, u_2(R) = z \\ G(z) = u_2(D/2) + \frac{\alpha}{k}(T_W - u_1(D/2)) = 0 \end{cases}$$

- Bestäm z så att villkoret víd $r = D/2$ blir uppfyllt: $G(z) = 0$
- Lös ODE systemet med gissade z som IVP med t ex ODE45
- Iterera med t ex Newton-Raphson
  - 1 iteration ska räcka, lineärt problem
  - Hur beräkna Jacobi matris dG/dz ? – numerisk derivering!

Randvärdesproblem, inskjutningsmetod, MATLAB I

$$\begin{cases} u_1' = u_2, \qquad u_1(R) = T_B \\ u_2' = -u_2/r, u_2(R) = z \\ G(z) = u_2(D/2) + \dfrac{\alpha}{k}(T_W - u_1(D/2)) = 0 \end{cases}$$

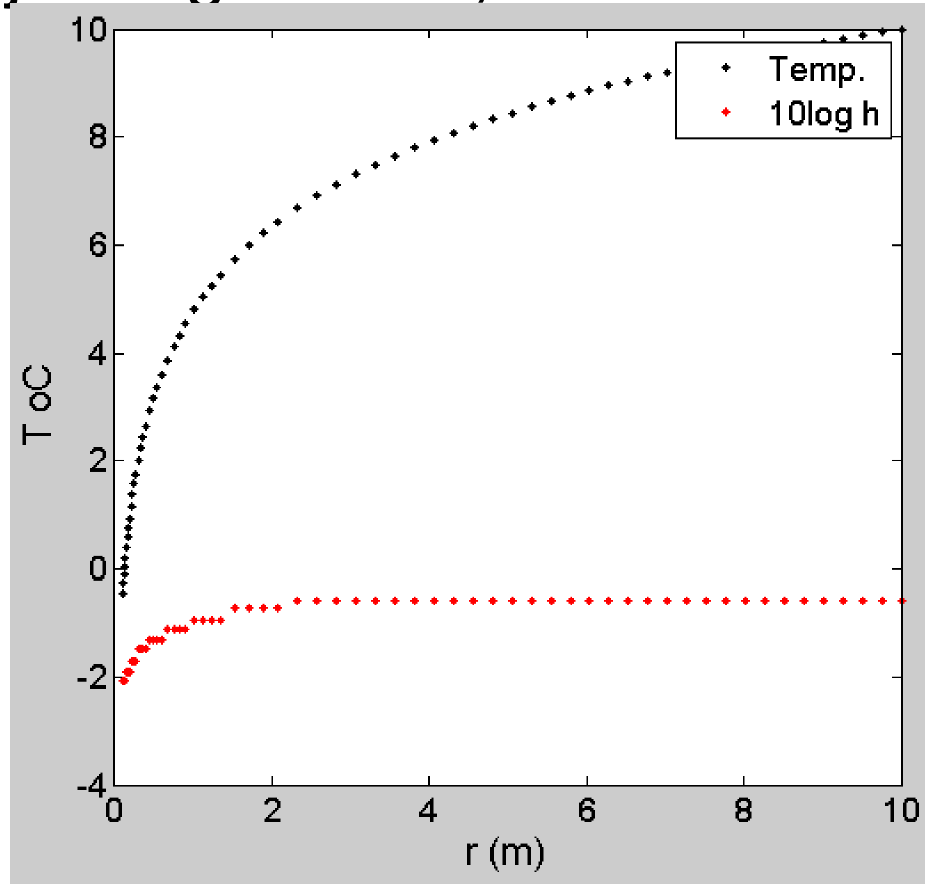
```
% temp-BVP med inskjutning
clear all, close all
global TW TB D k alfa R opts
D = 0.2; k = 20; alfa = 100;
R = 10; TW = -5; TB = 10;
tol  = 1e-6; z0 = 1;
z    = NewRaph('Gtemp',z0,tol);
opts = odeset('reltol',1e-4,'abstol',1e-4);
[rout,uout]=
ode45('tempODE',[R D/2],[TB; z],opts);

function r = Gtemp(z)
global TW TB D k alfa R opts
[rout,uout]=…
ode45('tempODE',[R D/2],[TB; z],opts);
TD2  = uout(end,1); TrD2 = uout(end,2);
r    = TrD2 + alfa/k*(TW-TD2);
```

```
function x = NewRaph(F,x,tol)
h    = 2*tol*norm(x);
steg = 1e-5;
iter = 0; itmax = 20;
while norm(h) > tol*norm(x)
   f = feval(F,x);
   J = jacobian(F,x,steg);
   h = J\f; x = x-h;
   iter = iter + 1;
   if iter > itmax
       break
   end
end
```

Randvärdesproblem, inskjutningsmetod, MATLAB II

- Fel $O(h^4)$, 65 steg ☺
- Steglängdsminskning vid $r = D/2$ ☺ Differens-metoden också (?)
- Inskjutning över långa intervall svårt ☹

```
        z          zn - zn-1
    0.2271        0.7729
    0.2271        0.0000
    0.2271        0.0000
P =
  285.3750
Pex =
  285.3758

z0 = 1;
EN iteration, sedan klart
```



$$T(r) = c_1 + c_2 \ln r \Rightarrow$$

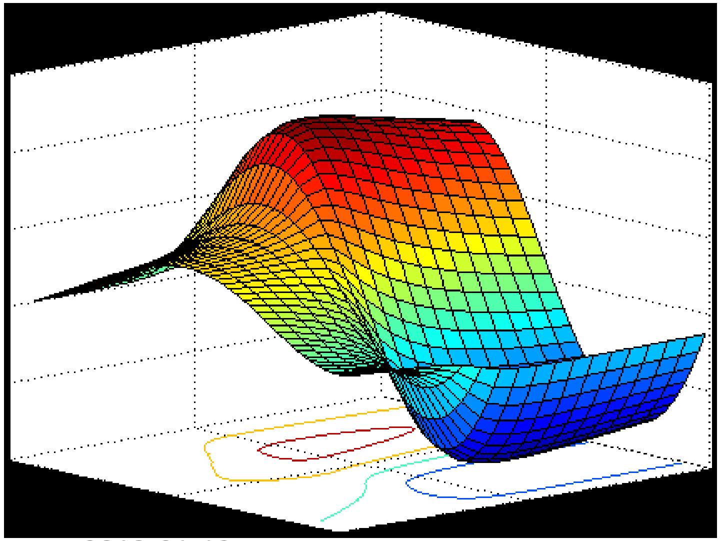
$$P_{exact} = \frac{2\pi k(T_W - T_B)}{\ln(\frac{D}{2R}) - \frac{2k}{D\alpha}}$$

Icke-linjärt randvärdesproblem. I

$$y'' + cy^2 - (6 - y)x = 0$$
$$y(0) = 1, y(4) = -1$$

- NAM p 119, differensmetod
- Parametersvep: c = 0, c1, c2,…
- c = 0 linjärt, lätt
- Fortsättning/Homotopi/…

Lösning för c_i tas som startvärde for c_{i+1}.



```
function [r,J] = fnam(u)
global X C H
d2u = diff(diff([1;u;-1]))/H/H;
r = d2u+C*u.^2-(6-u).*X;
if nargout > 1
    n = length(X);
    J = spdiags(ones(n,1)*[1 -2 1],…
        -1:1,n,n)/H/H;
    J = J + 2*C*diag(u) + diag(X);
end
%===========
global X C H
…
for C = clist
    k = k+1;
    u = NewRaph('fnam',u,1e-5);
    utab(k,:)=u;
end
```