

F8

- Randvärdesproblem NAM 8 , exempel
- Differensmetod
- Inskjutning
- Formuleringar

Randvärdesproblem, I

Exempel: Hur stor effekt per m borrhål?

Temperaturfördelning runt låångt bergvärmeborrhål:

Fourier's lag och axialsymmetri:

$$\rho C \frac{\partial T}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left(r k \frac{\partial T}{\partial r} \right)$$

Differentialekvation

$$T(\infty, t) = T_B$$

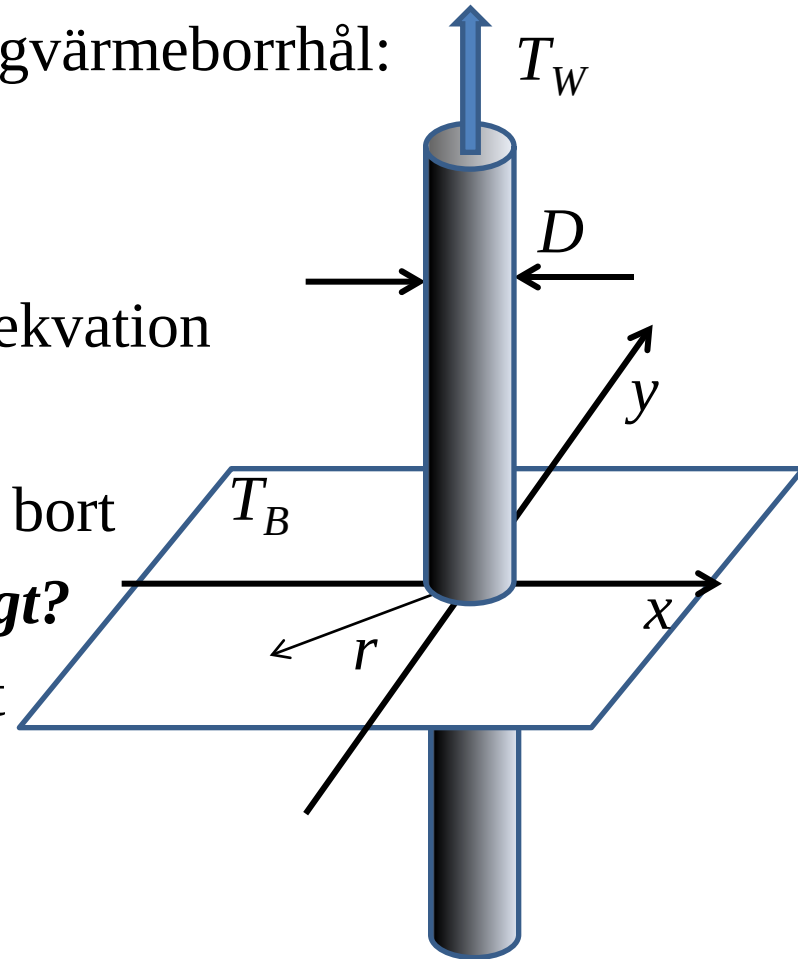
Temp. långt bort

$$k \frac{\partial T}{\partial r} \bigg|_{r=D/2} = -\alpha (T_W - T) \quad \text{2D! rimligt?}$$

Värmeövergång, vattnet

$$P = \pi D k \frac{\partial T}{\partial r} \bigg|_{r=D/2}$$

Effekt



Randvärdesproblem, differensmetod, I

Stationärt: $d/dt = 0$

Linjärt problem "AT=b"

lokalt fel $O(h^2)$

$$i = 1, 2, \dots, N-1$$

$$\frac{dT}{dr}(r = r_i) \rightarrow \frac{T_{i+1} - T_{i-1}}{2h}$$

$$\frac{d^2T}{dr^2}(r = r_i) \rightarrow \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}$$

$$T_N = T_B$$

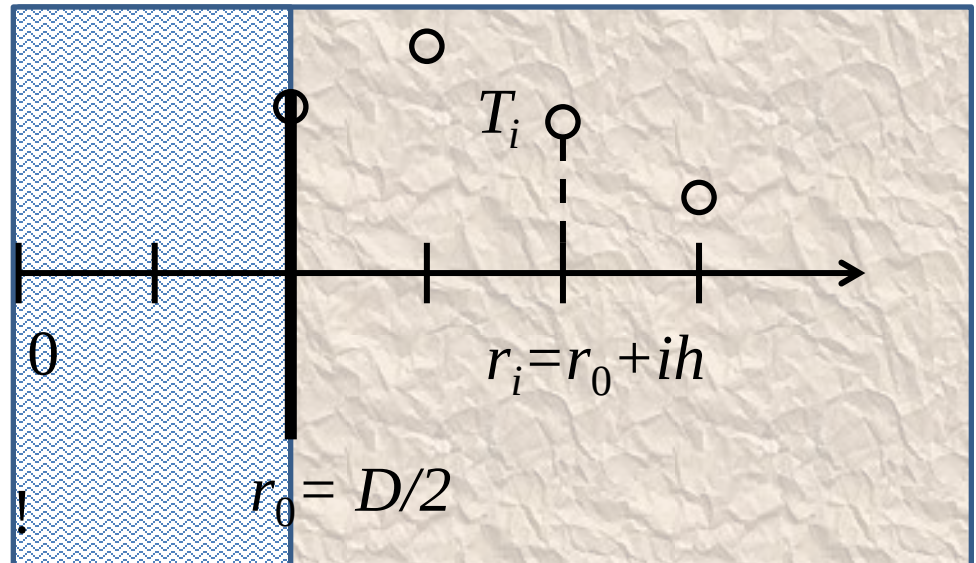
$$\left. \frac{dT}{dr} \right|_{r=D/2} = -\frac{\alpha}{k}(T_W - T) \rightarrow$$

$$\frac{T_1 - T_0}{h} + \frac{\alpha}{k}(T_W - T_0) = 0 \quad \textbf{Fel } O(h) !$$

$$\frac{1}{r} \frac{d}{dr} \left(rk \frac{dT}{dr} \right) = 0 : \frac{dT}{dr} + r \frac{d^2T}{dr^2} = 0$$

$$T(R) = T_B$$

$$\left. \frac{dT}{dr} \right|_{r=D/2} = -\frac{\alpha}{k}(T_W - T)$$



Randvärdesproblem, differensmetod II

$$\frac{T_1 - T_0}{h} + \frac{\alpha}{k}(T_W - T_0) = 0:$$

$$-\left(\frac{\alpha}{k} + \frac{1}{h}\right)T_0 + \frac{1}{h}T_1 = -\frac{\alpha}{k}T_W, i=0$$

$$\frac{T_{i+1} - T_{i-1}}{2h} + r_i \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} = 0:$$

$$\left(-\frac{1}{2h} + \frac{r_i}{h^2}\right)T_{i-1} + \left(-\frac{2r_i}{h^2}\right)T_i + \left(\frac{1}{2h} + \frac{r_i}{h^2}\right)T_{i+1} = 0,$$

$$T_N = T_B, i = N$$

```
N = 8;          % T(1...N+1)
r = linspace(D/2,R,N+1);
h = r(2)-r(1); h2 = h*h;
A = zeros(N+1,N+1);
b = zeros(N+1,1);
```

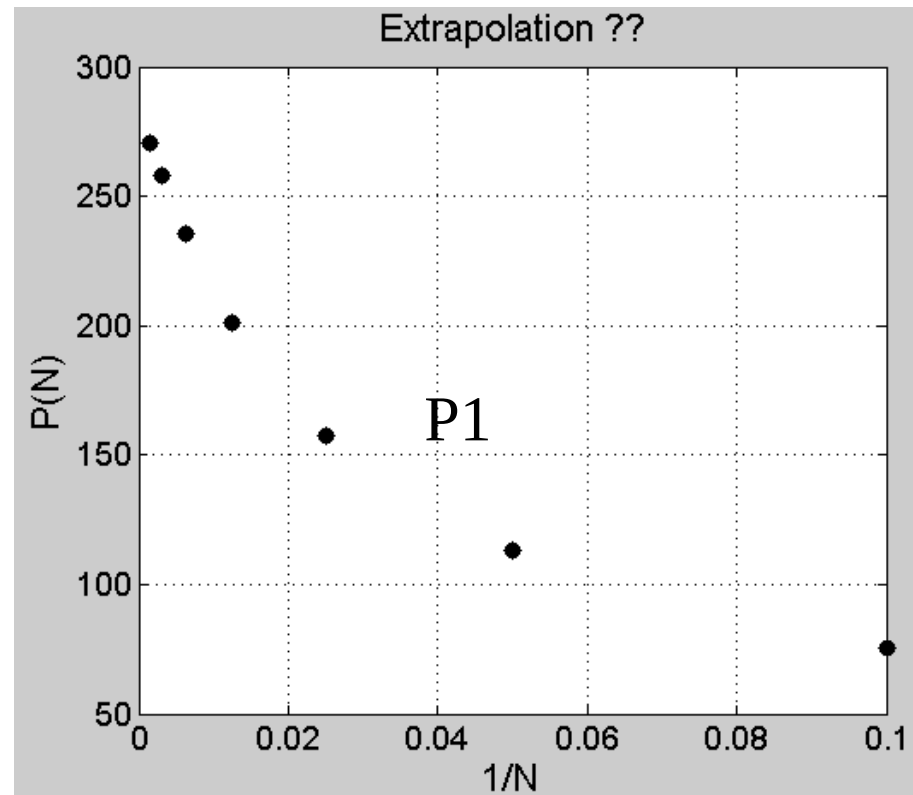
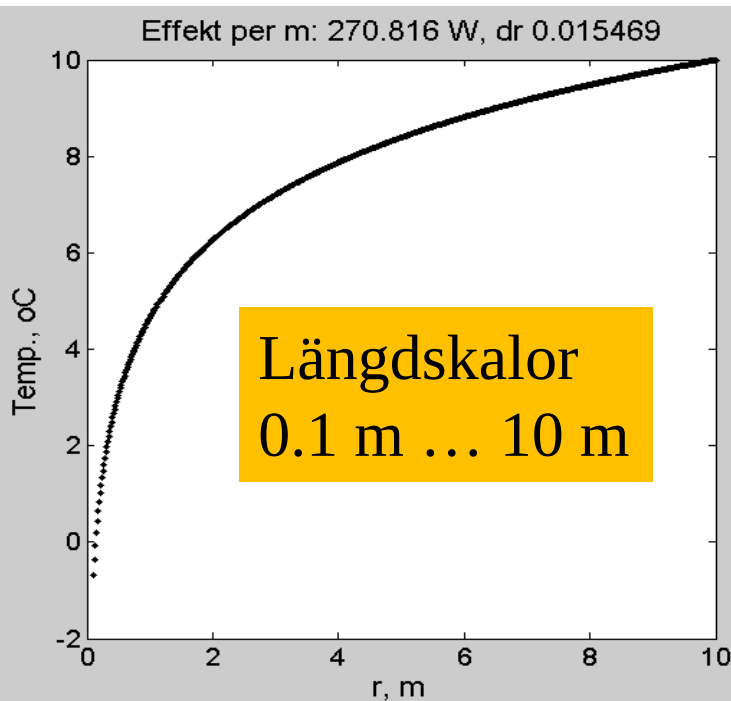
```
A(1,1) = -(alfa/k+1/h);
A(1,2) = 1/h;
b(1)    = -alfa/k*TW;
```

```
for j = 2:N % k = i+1
    rj = r(j);
    A(j,j-1) = -1/(2*h)+rj/h2;
    A(j, j) = -2*rj/h2;
    A(j,j+1) = 1/(2*h)+rj/h2;
end
```

```
A(N+1,N+1) = 1;
b(N+1) = TB;
```

Randvärdesproblem, differensmetod III

N	P1	P2
10	75.7	
20	113.1	141.4
40	157.8	190.8
80	201.2	234.0
160	235.3	262.6
320	257.8	277.0
640	270.8	282.7
1280	277.9	284.6



Fel $O(h)$... oekonomiskt ?
Förbättra approx. av dT/dr
i randvillkoret (P2)

$$\frac{dT}{dr}(r_0) \rightarrow (-3/2 \cdot T_0 + 2 \cdot T_1 - 1/2 \cdot T_2)/h, \text{ Fel } O(h^2)$$

Randvärdesproblem, differensmetod II

```
function res = tempbvpres(T)
global TW TB D k alfa R
n = length(T);
res = 0*T;
r = linspace(D/2,R,n)';
h = r(2)-r(1);
h2 = h*h;
% BVC at r = D/2 T(1)
res(1) = (T(2)-T(1))/h + ...
alfa/k*(TW-T(1));
% the inner points
D2T = diff(diff(T));
res(2:n-1) = (T(3:n)-T(1:n-2))/(2*h) +
             r(2:n-1).*D2T/h2;
% BVC at r = R
res(n) = T(n)-TB;
%=====
N = 80;
b = -tempbvpres(zeros(N,1));
A = jacobian('tempbvpres',zeros(N,1),1e-4); T = A \ b;
```

Lös ekv. syst. $\mathbf{res} = \mathbf{AT} - \mathbf{b} = \mathbf{0}$
 Jacobian-matrisen kan konstrueras
 med numerisk differentiering
 som i Newtons metod

$$res(1) = \frac{T_2 - T_1}{h} + \frac{\alpha}{k}(T_W - T_1)$$

$$res(i) = \frac{T_{i+1} - T_{i-1}}{2h} + \dots + r_i \frac{T_{i+1} - 2T_i + T_{i-1}}{h^2}, i = 2, \dots, n-1$$

$$res(n) = T_n - T_B$$

Randvärdesproblem, inskjutningsmetod I

$$\frac{dT}{dr} + r \frac{d^2 T}{dr^2} = 0, T(R) = T_B, \left. \frac{dT}{dr} \right|_{r=D/2} = -\frac{\alpha}{k}(T_W - T)$$

$\Rightarrow$

$$\begin{cases} u_1' = u_2, & u_1(R) = T_B \\ u_2' = -u_2 / r, & u_2(R) = z \\ G(z) = u_2(D/2) + \frac{\alpha}{k}(T_W - u_1(D/2)) = 0 \end{cases}$$

- Bestäm z så att villkoret vid $r = D/2$ blir uppfyllt: $G(z) = 0$
- Lös ODE systemet med gissade z som IVP med t ex ODE45
- Iterera med t ex Newton-Raphson
 - 1 iteration ska räcka, lineärt problem
 - Hur beräkna Jacobi matris dG/dz ? – numerisk derivering!

Randvärdesproblem, inskjutningsmetod, MATLAB I

$$\begin{cases} u_1' = u_2, & u_1(R) = T_B \\ u_2' = -u_2 / r, & u_2(R) = z \\ G(z) = u_2(D/2) + \frac{\alpha}{k}(T_W - u_1(D/2)) = 0 \end{cases}$$

```
% temp-BVP med inskjutning
clear all, close all
global TW TB D k alfa R opts
D = 0.2; k = 20; alfa = 100;
R = 10; TW = -5; TB = 10;
tol = 1e-6; z0 = 1;
z = NewRaph('Gtemp',z0,tol);
opts = odeset('reltol',1e-4,'abstol',1e-4);
[rou t,uout]=
ode45('tempODE',[R D/2],[TB; z],opts);
```

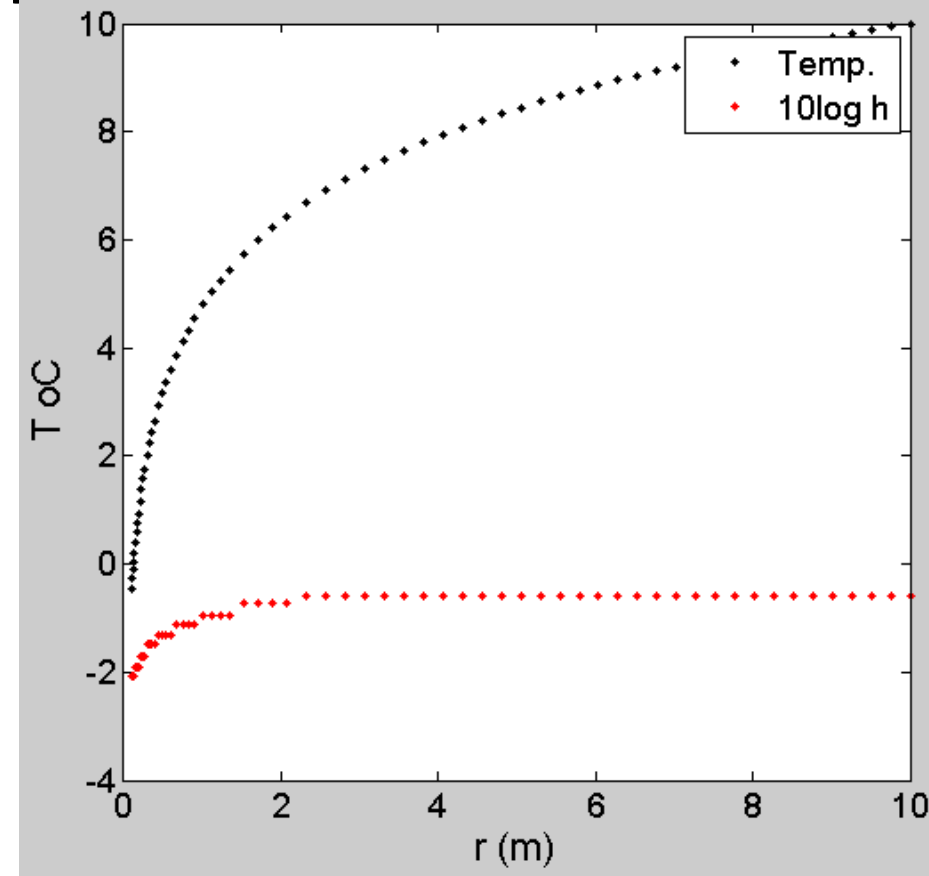
```
function r = Gtemp(z)
global TW TB D k alfa R opts
[rou t,uout]=...
ode45('tempODE',[R D/2],[TB; z],opts);
TD2 = uout(end,1); TrD2 = uout(end,2);
r = TrD2 + alfa/k*(TW-TD2);
```

```
function x = NewRaph(F,x,tol)
h = 2*tol*norm(x);
steg = 1e-5;
iter = 0; itmax = 20;
while norm(h) > tol*norm(x)
    f = feval(F,x);
    J = jacobian(F,x,steg);
    h = J\f; x = x-h;
    iter = iter + 1;
    if iter > itmax
        break
    end
end
```


Randvärdesproblem, inskjutningsmetod, MATLAB II

- Fel $O(h^4)$, 65 steg ☺
- Steglängdsminskning vid $r = D/2$ ☺ Differens-metoden också (?)
- Inskjutning över långa intervall svårt ☹

z	zn - zn-1
0.2271	0.7729
0.2271	0.0000
0.2271	0.0000
P =	
285.3750	
Pex =	
285.3758	
z0 = 1;	
EN iteration, sedan klart	



$$T(r) = c_1 + c_2 \ln r \Rightarrow$$

$$P_{exact} = \frac{2\pi k(T_W - T_B)}{\ln\left(\frac{D}{2R}\right) - \frac{2k}{D\alpha}}$$

Icke-linjärt randvärdesproblem. I

- NAM p 119, differensmetod
 - Parametersvep: $c = 0, c_1, c_2, \dots$
 - $c = 0$ linjärt, lätt
 - Fortsättning/Homotopi/...
- Lösning för c_i tas som startvärde för c_{i+1} .

$$y'' + cy^2 - (6 - y)x = 0$$
$$y(0) = 1, y(4) = -1$$

```
function [r,J] = fnam(u)
global X C H
d2u = diff(diff([1;u;-1]))/H/H;
r = d2u+C*u.^2-(6-u).*X;
if nargin > 1
    n = length(X);
    J = spdiags(ones(n,1)*[1 -2 1],...
        -1:1,n,n)/H/H;
    J = J + 2*C*diag(u) + diag(X);
end
%=====
global X C H
...
for C = clist
    k = k+1;
    u = NewRaph('fnam',u,1e-5);
    utab(k,:) = u;
end
```

