DN2222
Applied Numerical Methods
- part 2:
Numerical Linear Algebra

Lecture 2: Error Analysis & Gaussian Elimination

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Norm

- A norm is a function mapping from R^n to R fulfilling the following three conditions:
- $f(x) \geq 0$ and $f(x) = 0$ if and only if $x = 0$.
- $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \epsilon R$ and all $x \epsilon R^n$
- $f(x+y) \leq f(x) + f(y)$ for all x and $y \epsilon R^n$
- A norm is often denoted by $|| \cdot ||$ (avoid $| \cdot |$!!!)

Famous Norms

- Euclidian norm, $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- Infimum norm, $||x||_\infty = \max_{i=1:n} |x_i|$
- Maximum norm (or 1-norm), $||x||_1 = \sum_{i=1}^{n} |x_i|$
- ℓ_p-norm, $||x||_p = (\sum_{i=1}^{n} x_i^p)^{(1/p)}$

Quiz

- If $||x||_2 < ||y||_2$ what about the relation between $||x||_\infty$ and $||y||_\infty$?

Quiz' answer

- If $||x||_2 < ||y||_2$ what about the relation between $||x||_\infty$ and $||y||_\infty$?
- Nothing! See below:
- Ex1: If $x = (1, 2)$ and $y = (3, 4)$ then $2 = ||x||_\infty < ||y||_\infty = 4$
- Ex2: If $x = (10, 1)$ and $y = (8, 9)$ then $10 = ||x||_\infty > ||y||_\infty = 9$

Equivalent Norms

- Two norms $||\cdot||_\alpha$ and $||\cdot||_\beta$ in R^n are *equivalent* if there exists two constants $m > 0$ and $M > 0$ such that for all $x \epsilon R$

$$m||x||_\alpha \leq ||x||_\beta \leq M||x||_\alpha$$

- All vector norms in R^n are equivalent

Equivalent Norms (cont.)

The following relations are useful when converting error bounds in terms of one norm to error bounds in terms of another norm.

$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$$

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

$$||x||_\infty \leq ||x||_1 \leq n||x||_\infty$$

(D p21: Lemma 1.5)

Inner product

Let B be a complex linear space. $< \cdot, \cdot >: B \times B \to R$ is an inner product if all of the follwing apply:

1) $< x, y >= < \bar{y, x} >$
2) $< x, y + z >=< x, y > + < x, z >$
3) $< \alpha x, y >= |\alpha| < x, y >$
4) $< x, x >\geq 0 < x, x >= 0$ iff $x = 0$

- x and y are orthogonal if $< x, y >= 0$
- $\sqrt{< x, x >}$ is a norm.

Cauchy-Schwartz' inequality

- $| < x, y > | \leq \sqrt{< x, x > \cdot < y, y >}$

Induced matrix norm

Let A be $m \times n$, $||\cdot||_{\hat{m}}$ be a norm on R^m and $||\cdot||_{\hat{n}}$ be a norm on R^n. Then

$$||A||_{\hat{m}\hat{n}} = \max_{x \neq 0} \frac{||Ax||_{\hat{m}}}{||x||_{\hat{n}}}$$

is called an operator norm or *induced norm* or subordinate norm.

- For operator norms we have $||AB|| \leq ||A|| \cdot ||B||$.

Special matrix norms

- $||A||_1 = ||A^T||_\infty = \max_j \sum_i |a_{ij}| =$ maximum column sum.
- $||A||_2 = \sqrt{\lambda_{max}(A * A)}$
- $||A||_2 = ||A^T||_2$
- $||A||_F = \sqrt{\sum_i \sum_j |a_{ij}|^2}$, the Frobenius norm.

Perturbation Theory

Suppose $Ax = b$ and $(A + \delta A)\hat{x} = b + \delta b$. Our goal is to bound the norm of $\delta x = \hat{x} - x$.

$$\begin{aligned} (A + \delta A)(x + \delta x) &= b + \delta b \\ Ax &= b \\ \longrightarrow \quad \delta Ax + (A + \delta A)\delta x &= \delta b \end{aligned}$$

giving $\delta x = A^{-1}(\delta b - \delta A \hat{x})$ and

$$||\delta x|| \leq ||A^{-1}||(||\delta b|| + ||\delta A|| \cdot ||\hat{x}||)$$

We can also derive

$$\frac{||\delta x||}{||\hat{x}||} \leq ||A^{-1}|| \cdot ||A|| \cdot \left(\frac{||\delta A||}{||A||} + \frac{||\delta b||}{||A|| \cdot ||\hat{x}||} \right)$$

- The quantity $\kappa(A) = ||A^{-1}|| \cdot ||A||$ is the *condition number*.

Use the residual

$$\begin{aligned} Ax &= b \\ A\hat{x} &= \hat{b} \\ A(x - \hat{x}) &= b - \hat{b} \\ x - \hat{x} &= A^{-1}(b - \hat{b}) \\ x - \hat{x} &= A^{-1}(b - A\hat{x}) \\ x - \hat{x} &= A^{-1} r \\ ||x - \hat{x}|| &= ||A^{-1}|| \cdot ||r|| \end{aligned}$$

LU-decomposition

- There exists a unique unit lower triangular matrix L and a non-singular upper triangular matrix U such that $A = LU$.
- **D: Thm 2.5** If A is non-singular, there exist permutations P_1 and P_2, a unit lower matrix L, and a nonsingular upper triangular matrix U such that $P_1 A P_2 = LU$.
- one of the permutation matrices is neccessary. $P_1 A$ reorders the rows of A, AP_2 reorders the columns of A,

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Permutation matrix

- A permutation matrix is the identity-matrix with permuted rows.
- PX is the same as X with its rows permuted.
- XP is the same as X with its columns permuted.
- $P^{-1} = P^T$
- $\texttt{det}(P) = \pm 1$
- $P = \pm P_1 \cdot P_2$ is also a permutation matrix.

Floating Point Arithmetic

$$\text{fl}(a \circ b) = (a \circ b) \cdot (1 + \delta)$$

δ is bounded by ϵ, the machine epsilon or machine precision.

Example

- Calculate $z = a + b + c$
- Step 1: $w = \text{fl}(a + b) = (a + b) \cdot (1 + \delta)$
- Step 2: $z = \text{fl}(w + c) = (w + c) \cdot (1 + \delta)$
  $= ((a + b) \cdot (1 + \delta) + c) \cdot (1 + \delta)$
  $= (a + b + c) \cdot (1 + \delta) + (a + b) \cdot (1 + \delta)^2$
- Note that the error does not depend equally on all a, b, and c!!!

Gaussian Elimination

on the blackboard...

Pivoting

on the blackboard...

Some review questions:

- **Q21.** Define the condition number of a matrix.
- **Q22.** Describe what happens with the solution x of a linear system $Ax = b$ when the right hand side b is perturbed into $b + \delta b$.
- **Q25.** What is meant with pivoting for stability?
- **Q28.** How many arithmetic operations are needed for computing the Gaussian elimination LU-factorization of an $n \times n$-matrix A?
- **Q29.** How many arithmetic operations are needed to solve a linear system by forward and backward substitution, once the triangular factors L and U are computed?
- **Q31.** How many arithmetic operations are needed to multiply a $m \times n$ matrix by a $n \times p$ matrix? Compare with the count for Gaussian elimination!