# DN2222
# Applied Numerical Methods
# - part 2:
# Numerical Linear Algebra

## Lecture 2: Error Analysis & Gaussian Elimination

**2012-10-24**

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### Register for the course!

- Go to `https://rapp.csc.kth.se/` and activate yourself!
- Go to *Mina Sidor* and activate yourself!
- ...or you cannot register for the written exam Dec $10^{th}$!
- On-line registration for the exam is done using *Mina Sidor.* It is **only open** from (about) November $9^{th}$ to (about) November $23^{rd}$.
- **Exam registration only open mid November.**
- **No late registration for the exam allowed.**

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### Norm

- A norm is a function mapping from $R^n$ to $R$ fulfilling the following three conditions:
- $f(x) \geq 0$ and $f(x) = 0$ if and only if $x = 0$.
- $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \epsilon R$ and all $x \epsilon R^n$
- $f(x + y) \leq f(x) + f(y)$ for all $x$ and $y \epsilon R^n$
- A norm is often denoted by $|| \cdot ||$ (avoid $| \cdot |$ !!!)

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### Famous Norms

- Euclidian norm, $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$
- Infimum norm (or max norm), $||x||_\infty = \max_{i=1:n} |x_i|$
- Maximum norm (or 1-norm), $||x||_1 = \sum_{i=1}^{n} |x_i|$
- $\ell_p$-norm, $||x||_p = (\sum_{i=1}^{n} x_i^p)^{(1/p)}$

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### Equivalent Norms

- Two norms $||\cdot||_\alpha$ and $||\cdot||_\beta$ in $R^n$ are *equivalent* if there exists two constants $m > 0$ and $M > 0$ such that for all $x \epsilon R$

$$m||x||_\alpha \leq ||x||_\beta \leq M||x||_\alpha$$

- All vector norms in $R^n$ are equivalent

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### Equivalent Norms (cont.)

The following relations are useful when converting error bounds in terms of one norm to error bounds in terms of another norm.

$$||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$$

$$||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$$

$$||x||_\infty \leq ||x||_1 \leq n||x||_\infty$$

(D p21: Lemma 1.5)

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**Quiz**

- ◦ Find an array for which $||x||_\infty < ||x||_1$
- ◦ Find an array for which $||x||_\infty = ||x||_1$
- ◦ Find an array for which $||x||_1 = n||x||_\infty$

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**Quiz**

- If $||x||_2 < ||y||_2$ what about the relation between $||x||_\infty$ and $||y||_\infty$?

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**Quiz' answer**

- If $||x||_2 < ||y||_2$ what about the relation between $||x||_\infty$ and $||y||_\infty$?

⋆ Nothing! Since:

- ◦ Ex1: If $x = (1, 2)$ and $y = (3, 4)$ then $||x||_2 < ||y||_2$ and $||x||_\infty < ||y||_\infty$
- ◦ Ex2: If $x = (10, 1)$ and $y = (8, 9)$ then $||x||_2 < ||y||_2$ but $||x||_\infty > ||y||_\infty$

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**Inner product**

Let $B$ be a complex linear space. $< \cdot, \cdot >: B \times B \to R$ is an inner product if all of the follwing apply:

1) $< x, y > = \bar{< y, x >}$
2) $< x, y + z > = < x, y > + < x, z >$
3) $< \alpha x, y > = |\alpha| < x, y >$
4) $< x, x > \geq 0$ with $< x, x >= 0$ iff $x = 0$

- $x$ and $y$ are orthogonal if $< x, y > = 0$
- $\sqrt{< x, x >}$ is a norm.

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**Cauchy-Schwartz' inequality**

- $| < x, y > | \leq \sqrt{< x, x > \cdot < y, y >}$

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**Induced matrix norm**

Let $A$ be $m \times n$, $||\cdot||_{\hat{m}}$ be a norm on $R^m$ and $||\cdot||_{\hat{n}}$ be a norm on $R^n$. Then

$$||A||_{\hat{m}\hat{n}} = \max_{x\neq 0} \frac{||Ax||_{\hat{m}}}{||x||_{\hat{n}}}$$

is called an operator norm or *induced norm* or subordinate norm.

- For operator norms we have $||AB|| \leq ||A|| \cdot ||B||$.

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**Special matrix norms**

- $||A||_1 = ||A^T||_\infty = \max_j \sum_i |a_{ij}|$ = maximum column sum.
- $||A||_\infty = ||A^T||_1 = \max_i \sum_j |a_{ij}|$ = maximum row sum.
- $||A||_2 = \sqrt{\lambda_{max}(A * A)}$
- $||A||_2 = ||A^T||_2$
- $||A||_F = ||A^T||_F = \sqrt{\sum_i \sum_j |a_{ij}|^2}$, the Frobenius norm.

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**Perturbation Theory**

Suppose $Ax = b$ and $(A + \delta A)\hat{x} = b + \delta b$. Our goal is to bound the norm of $\delta x = \hat{x} - x$.

$$\begin{aligned} (A + \delta A)(x + \delta x) &= b + \delta b \\ Ax &= b \\ \longrightarrow \quad \delta A x + (A + \delta A)\delta x &= \delta b \end{aligned}$$

giving $\delta x = A^{-1}(\delta b - \delta A \hat{x})$ and

$$||\delta x|| \leq ||A^{-1}||(||\delta b|| + ||\delta A|| \cdot ||\hat{x}||)$$

We can also derive

$$\frac{||\delta x||}{||\hat{x}||} \leq ||A^{-1}|| \cdot ||A|| \cdot \left( \frac{||\delta A||}{||A||} + \frac{||\delta b||}{||A|| \cdot ||\hat{x}||} \right)$$

$$= \{\text{D p33}\} =\leq \frac{\kappa(A)}{1 - \kappa(A)\frac{||\delta A||}{||A||}} \left( \frac{||\delta A||}{||A||} + \frac{||\delta b||}{||b||} \right)$$

- The quantity $\kappa(A) = ||A^{-1}|| \cdot ||A||$ is the *condition number.*

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**Condition number,**

- **Theorem 2.1** (p33) Let A be non-singular then

  $$\min\{||\delta A||_2 : A + \delta A \; is \; singular\} = \frac{1}{||A^{-1}|| \cdot ||A||} = \frac{1}{\kappa(A)}$$

- The distance from a non-singular matrix A to the nearest singular matrix is $\frac{1}{\kappa(A)}$.

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**Use the residual**

$$\begin{aligned} Ax &= b \\ A\hat{x} &= \hat{b} \\ A(x - \hat{x}) &= b - \hat{b} \\ x - \hat{x} &= A^{-1}(b - \hat{b}) \\ x - \hat{x} &= A^{-1}(b - A\hat{x}) \\ x - \hat{x} &= A^{-1} r \end{aligned}$$

$$||x - \hat{x}|| \leq ||A^{-1}|| \cdot ||r||$$

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### LU-decomposition

- There exists a unique unit lower triangular matrix $L$ and a non-singular upper triangular matrix $U$ such that $A = LU$.
- **D: Thm 2.5** If $A$ is non-singular, there exist permutations $P_1$ and $P_2$, a unit lower matrix $L$, and a nonsingular upper triangular matrix $U$ such that $P_1 A P_2 = LU$.
- Only one of the permutation matrices is neccessary. $P_1 A$ reorders the rows of $A$, $AP_2$ reorders the columns of $A$.

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### Permutation matrix

- A permutation matrix is the identity-matrix with permuted rows.
- $PX$ is the same as $X$ with its rows permuted.
- $XP$ is the same as $X$ with its columns permuted.
- $P^{-1} = P^T$
- $\texttt{det}(P) = \pm 1$

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### Quiz

◦ If $P_1$ and $P_2$ are permutation matrices.
Let $P = P_1 \cdot P_2$.
Is $P$ a permutation matrix?

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### Quiz

◦ If $P_1$ and $P_2$ are permutation matrices.
Let $P = P_1 \cdot P_2$.
Is $P$ a permutation matrix?

⋆ Yes:

- If $P_1$ and $P_2$ are permutation matrices. $P = \pm P_1 \cdot P_2$ is also a permutation matrix.

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### Floating Point Arithmetic

$$\text{fl}(a \circ b) = (a \circ b) \cdot (1 + \delta)$$

$\delta$ is bounded by $\epsilon$, the machine epsilon or machine precision.

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### Example

- Calculate $z = a + b + c$
- Step 1: $w = \text{fl}(a + b) = (a + b) \cdot (1 + \delta_1)$
- Step 2: $z = \text{fl}(w + c) = (w + c) \cdot (1 + \delta_2)$
  $= ((a + b) \cdot (1 + \delta_1) + c) \cdot (1 + \delta_2)$

$$\begin{aligned}
&= ((a+b)\cdot(1+\delta_1)\cdot(1+\delta_2) + c\cdot(1+\delta_2) \\
&= \{\delta_i \le \epsilon\} \\
&= (a+b)\cdot(1+\epsilon)^2 + c\cdot(1+\epsilon)
\end{aligned}$$

- Note that the error does not depend equally on all $a$, $b$, and $c$!!!

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**Gaussian Elimination**

on the blackboard...

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**Pivoting**

on the blackboard...

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**GEPP**

- The main empirical observation, justified by decades of experience, is that GEPP almost always keeps $||L||\cdot||U|| \approx ||A||$.
- We define the pivot growth factor for GEPP as $g_{pp} = ||U||_\infty/||A||_\infty$

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**Some practical error estimates:** (D p54)

$$\text{error} = \frac{||\hat{x}-x||_\infty}{||\hat{x}||} \le ||A^{-1}||\cdot\frac{||r||}{||\hat{x}||}$$

$$\text{error} = \frac{||\hat{x}-x||_\infty}{||\hat{x}||} \le ||\frac{|A^{-1}|\cdot|r|}{||\hat{x}||}||$$

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**Some review questions:**

- **Q21.** Define the condition number of a matrix.
- **Q22.** Describe what happens with the solution $x$ of a linear system $Ax = b$ when the right hand side $b$ is perturbed into $b + \delta b$.
- **Q25.** What is meant with pivoting for stability?
- **Q28.** How many arithmetic operations are needed for computing the Gaussian elimination $LU$-factorization of an $n\times n$-matrix A?
- **Q29.** How many arithmetic operations are needed to solve a linear system by forward and backward substitution, once the triangular factors $L$ and $U$ are computed?
- **Q31.** How many arithmetic operations are needed to multiply a $m\times n$ matrix by a $n\times p$ matrix? Compare with the count for Gaussian elimination!

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