Efficient Discrete Laplacian in Matlab

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Consider the discretization of the Laplacian $-\Delta$ subject to homogeneous Dirichlet boundary conditions over the unit cube $\Omega = [0,1]^d$, for $d = 1,2,3$. Let N be an integer and $h = (N+1)^{-1}$. In every coordinate direction, we use centered second order finite differences.

- $d = 1$: If the discrete unknowns are denoted by $x_i = ih$, the discrete negative Laplacian is given by

$$T_{1,N} = \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix}.$$

  Here, we assumed the unknowns corresponding to the boundary conditions to be eliminated.

- $d = 2$: In that case, we obtain the standard five-point stencil

$$\begin{bmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{bmatrix}.$$

  Assume that the unknowns $(x_i, y_j) = (ih, jh)$ are ordered lexicographically. This gives rise to the well-known matrix $T_{2,N\times N}$. Let $\otimes$ denote the Kronecker product. Then this matrix has a very elegant representation:

$$T_{2,N\times N} = T_{1,N} \otimes I + I \otimes T_{1,N}.$$

  In Matlab, the Kronecker product is explicitely available via the command `kron`. Therefore, we can generate $T_{2,N\times N}$ rather easily:

```
T1N = 2*speye(N) - spdiags(ones(N-1,1),1,N,N) - spdiag(ones(N-1,1),-1,N,N)
T2NxN = kron(T1N,speye(N)) + kron(speye(N),T1N)
```

- $d = 3$: Assuming lexicographic ordering of the unknowns $(x_i, y_j, z_k) = (ih, jh, kh)$ we obtain similarly as above

$$T_{3,N\times N\times N} = T_{1,N} \otimes I \otimes I + I \otimes T_{1,N} \otimes I + I \otimes I \otimes T_{1,N}.$$