Homework 3, DN2230

Due December 1, 2010

If a correct solution is handed in before the deadline (i.e. December 1) two bonus points will be awarded to the final written exam. If a solution that is handed in before that date is not correct, it has to be redone, but the second time without yielding bonus points for the exam.

We are interested in solving partial differential equations in 3D. Consider the following two partial differential equations:

(a) $-\Delta u + 10^6 u_x = F$ with the exact solution $u(x,y,z) = xyz(1-x)(1-y)(1-z)$.

(b) $-\Delta u + 10^6 x^2 u_x + 1000u = F$ with the exact solution $u(x,y,z) = \exp(xyz)\sin(\pi x)\sin(\pi y)\sin(\pi z)$.

The computational domain is $\Omega = [0,1]^3$. The boundary conditions are homogeneous Dirichlet conditions, i.e. $u = 0$ on the boundary $\partial\Omega$. Start by computing the right-hand sides F (i.e. the continuous functions, using pen and paper) in (a) and (b) such that the given exact solution appears. In order to discretize the problem, standard stencils will be applied. Let $h = 1/12$ denote the stepsize in all directions. Then $-\Delta$ is replaced by the 7-point stencil, u_x by the symmetric 2nd-order difference.

Matlab contains implementations of the following iterative methods:

> BiCG, BiCGStab, CGS, GMRES, GMRES(k) (see Matlab's documentation for GMRES), QMR.

It also provides an implementation of different ILU preconditioners (luinc and cholinc).

For plotting the iteration history, it is appropriate to represent the residual in a logarithmic scale. Matlab provides the `semilogy` command for that purpose. Since we are interested in a comparision of the behaviour of different algorithms, curves representing related experiments should be drawn on one figure.

Note that it might be necessary to set the maximal number of iterations quite large, at least for some iterative methods and preconditioners.

PROBLEMS

Choose a start vector x_0 . This may be a vector consisting of only ones, zeros, or a randomly generated nonzero vector.

1. Testing of preconditioners: Choose a method. Plot the iteration history (i.e. plot the norm of the residual versus iteration number) using the following preconditioners: no preconditioning (identity matrix $M = I$), diagonal preconditioning ($M = \text{diag}(A)$), ILU(0) preconditioning (`luinc`(A,'0')), and ILU(θ) preconditioning with an appropriate θ (`luinc`(A,θ)). You could e.g. try a θ such that the number of nonzero elements in L and U are approximately four times larger than the number of nonzeros in A. You can use the command `nnz`.

   We know that the spectrum of the preconditioned matrix has an important influence on the behavior of the iteration methods. Therefore, plot additionally the eigenvalue distribution of the preconditioned matrices in the complex plane. Note: The command eig can only be applied to full matrices. Therefore, one should e.g. use `eig(full`(M2\(M1\A))) if using a preconditioner $M = M_1 M_2$ (such as $M = LU$).

2. Choose one preconditioner. Run all methods on the test examples and plot the iteration history, i.e. plot the norm of the residual versus iteration number. Compare the performance of the different methods.