Homework 2, DN2230

Due November 21, 2011

If a correct solution is handed in before the deadline (i.e. November 21) two bonus points will be awarded to the final written exam. If a solution that is handed in before that date is not correct, it has to be redone, but the second time without yielding bonus points for the exam.

Solutions must be clearly written, and easy to follow. If not, they will not generate bonus points, and must be redone.

1. The goal of this exercise is to prove the following theorem.

   Theorem. *Let A be any non-singular $m \times m$ matrix and b any vector of length m. The GMRES method finds the exact solution of $Ax = b$ in at most m steps (i.e. $r_n = 0$ for some $n \leq m$).*

   You will construct the proof by solving each of the following subproblems:

   (a) Show that if $h_{n+1,n} \neq 0$ for $n \leq m-1$ in the Arnoldi iteration (no Arnoldi breakdown), then $\mathcal{K}_m = \mathbb{C}^m$.

   (b) Under the assumption in (a), show that the m-th iterate of the GMRES method satisfies the equation.

   (c) Assume that at some n, $h_{n+1,n} = 0$ in the Arnoldi iteration (Arnoldi breakdown). Show that $\mathcal{K}_n$ is an invariant subspace of A, i.e. $Av \in \mathcal{K}_n$, for every $v \in \mathcal{K}_n$.

   (d) Under the assumption in (c), show that $AQ_n = Q_n H_n$, where Q_n and H_n are the matrices defined in equations (33.1) and (33.8) in NLA.

   (e) Under the assumption in (c), show that H_n is invertible. This may for instance be seen by proving that each eigenvalue of H_n is an eigenvalue of A. Hence 0 can not be an eigenvalue, so H_n must be invertible.

   (f) Under the assumption in (c), show that the solution x to the system of equations $Ax = b$ lies in $\mathcal{K}_n$. Conclude that GMRES has found the solution to $Ax = b$ in step n.

2. Exercise 38.5 in "Numerical Linear Algebra" (NLA).

3. Exercise 38.6 in NLA.