INVERSE RECONSTRUCTION

The goal of this excercise is to reconstruct an unknown heat conductivity $a^*(x)$ inside a wall located at $x \in [0,1]$. To reconstruct the conductivity we apply heat fluxes on each side and measure the resulting surface temperatures $u^*(0,t)$ and $u^*(1,t)$. We here assume that the temperature distribution in the wall can be modelled by the heat equation in one space dimension i.e.

$$u_t = (a(x)^2 u_x)_x, \quad \text{for } (x,t) \in (0,1) \times (0,1], \tag{1}$$

with initial data $u(x,0) = u_0$ and Neumann boundary values (heat flux)

$$a(0)^2 u_x(0,t) = q_0, \quad a(1)^2 u_x(1,t) = q_1. \tag{2}$$

To solve this inverse problem we formulate the minimization problem

$$\min_a \frac{1}{2} \int_0^1 \Big(u(0,t) - u^*(0,t)\Big)^2 + \Big(u(1,t) - u^*(1,t)\Big)^2 \, \mathrm{d}t \tag{3}$$

where u^* is our measured data and u solves (1) and (2).

A solution (a,u) to (3) will satisfy the following system of equations

$$\begin{aligned}
u_t &= (a^2 u_x)_x, \quad \text{for } (x,t) \in (0,1) \times (0,1], \\
u(x,0) &= u_0, \\
a(0)^2 u_x(0,t) &= q_0, \\
a(1)^2 u_x(1,t) &= q_1, \\
-p_t &= (a^2 p_x)_x, \quad \text{for } (x,t) \in (0,1) \times (0,1], \\
p(x,1) &= 0, \\
a(0)^2 p_x(0,t) &= u(0,t) - u^*(0,t), \\
a(1)^2 p_x(1,t) &= u^*(1,t) - u(1,t), \\
a(x) \int_0^1 u_x p_x \, \mathrm{d}t &= 0,
\end{aligned} \tag{4}$$

where p is a dual variable satisfying a backward equation. The last equation in (4) describes an optimality condition for a.

Question 0.1. *Derive* (4) *from* (3). *Hint: Differentiate the lagrangian*

$$\mathcal{L}(u,p,a) = \frac{1}{2} \int_0^1 (u(0,t) - u^*(0,t))^2 + (u(1,t) - u^*(1,t))^2 \, \mathrm{d}t + \int_0^1 \int_0^1 (u_t - (a^2 u_x)_x) p \, \mathrm{d}x \, \mathrm{d}t.$$

The finite element method applied to the spacial dimension gives the following semi-discretized system for the forward and backward equations

$$\begin{aligned}
M\mathbf{u}_t &= -S(\mathbf{a})\mathbf{u} + \mathbf{f}, \quad t \in (0,1] \\
\mathbf{u}(0) &= \mathbf{u}_0, \\
-M\mathbf{p}_t &= -S(\mathbf{a})\mathbf{p} + \mathbf{g}, \quad t \in [0,1) \\
\mathbf{p}(1) &= 0,
\end{aligned} \tag{5}$$

where S is the stiffness matrix, M is the mass matrix, and $\mathbf{f}$, $\mathbf{g}$ are load vectors that includes the boundary conditions. The stiffness matrix here depends on the conductivity distribution and the vectors $\mathbf{u}$, $\mathbf{p}$, $\mathbf{f}$, $\mathbf{g}$ depends on the time.

Question 0.2. *Solve system* (4) *with* $u_0 = 300$,

$$q_0 = q_1 = \begin{cases} 100\sin(5\pi t) & , 5\pi t < \pi \\ 0 & , 5\pi t \geq \pi \end{cases}$$

and with different (noisy) boundary measurements $u(0,t)$ *and* $u(1,t)$ *given in the files* `data0.mat`, `data1.mat` *and* `data2.mat`. *The file* `data0.mat` *contains measurements corresponding to* $a = 1$. *All measurements contains data from 1000 equidistant time steps so you have to interpolate them to fit your time discretization.*

The easiest way to solve (4) *is to start with an initial guess for the conductivity and solve* (5) *with a suitable Euler method. The conductivity can then be updated by taking a gradient step in* $a(x)$, *i.e.*

$$a_{n+1} = a_n - \theta a_n \int_0^1 u_x p_x \, \mathrm{d}t, \quad x \in [0,1]$$

where θ *is the step length. To your help you have the function* `assemble` *which assembles the siffness matrix, mass matrix, and load vector depending on the coefficient and the boundary conditions.*

Question 0.3. *Assume that* u *solves the stationary heat equation*

$$\begin{aligned} &(a^2 u_x)_x = 0, \quad x \in (0,1), \\ &u(0) = 0, \\ &a(1)^2 u_x(1) = q, \end{aligned}$$

and we want to reconstruct $a(x)$ *from the measurement* $u(1)$. *What can we say about this problem?*