Chapter 5

The Monte-Carlo Method

5.1 Statistical Error

This chapter gives the basic understanding of simulation of expected values $E[g(X(T))]$ for a solution, X, of a given stochastic differential equation with a given function g. In general the approximation error has the two parts of statistical error and time discretization error, which are analyzed in the next sections. The estimation of statistical error is based on the Central Limit Theorem. The error estimate for the time discretization error of the Euler method is directly related to the proof of Feyman-Kǎc's theorem with an additional residual term measuring the accuracy of the approximation, which turns out to be first order in contrast to the half order accuracy for strong approximation.

Consider the stochastic differential equation

$$dX(t) = a(t, X(t))dt + b(t, X(t))dW(t)$$

on $t_0 \leq t \leq T$, how can one compute the value $E[g(X(T))]$? The Monte-Carlo method is based on the approximation

$$E[g(X(T))] \simeq \sum_{j=1}^{N} \frac{g(\overline{X}(T;\omega_j))}{N},$$

where $\overline{X}$ is an approximation of X, e.g. the Euler method. The error in the

Monte-Carlo method is

$$
\begin{aligned}
& E[g(X(T))] - \sum_{j=1}^{N} \frac{g(\overline{X}(T;\omega_j))}{N} \\
&= E[g(X(T)) - g(\overline{X}(T))] - \sum_{j=1}^{N} \frac{g(\overline{X}(T;\omega_j)) - E[g(\overline{X}(T))]}{N}. \quad (5.1)
\end{aligned}
$$

In the right hand side of the error representation (5.1), the first part is the time discretization error, which we will consider in the next subsection, and the second part is the statistical error, which we study here.

Example 5.1 Compute the integral $I = \int_{[0,1]^d} f(x)dx$ by the Monte Carlo method, where we assume $f(x) : [0,1]^d \to \mathbf{R}$.

Solution. We have

$$
\begin{aligned}
I &= \int_{[0,1]^d} f(x)\ dx \\
&= \int_{[0,1]^d} f(x)p(x)\ dx \quad (\text{ where } p \text{ is the uniform density function}) \\
&= E[f(x)] \qquad\qquad (\text{ where } x \text{ is uniformly distributed in } [0,1]^d) \\
&\simeq \sum_{n=1}^{N} \frac{f(x(\omega_n))}{N} \\
&\equiv I_N,
\end{aligned}
$$

where $\{x(\omega_n)\}$ is sampled uniformly in the cube $[0,1]^d$, by sampling the components $x_i(\omega_n)$ independent and uniformly on the interval $[0,1]$. □

The Central Limit Theorem is the fundamental result to understand the statistical error of Monte Carlo methods.

Theorem 5.2 (The Central Limit Theorem) *Assume* ξ_n, $n = 1,2,3,\ldots$ *are independent, identically distributed (i.i.d) and* $E[\xi_n] = 0$, $E[\xi_n^2] = 1$. *Then*

$$
\sum_{n=1}^{N} \frac{\xi_n}{\sqrt{N}} \rightharpoonup \nu, \tag{5.2}
$$

where ν is $N(0,1)$ and $\rightharpoonup$ denotes convergence of the distributions, also called weak convergence, i.e. the convergence (5.2) means $E[g(\sum_{n=1}^N \xi_n/\sqrt{N})] \to E[g(\nu)]$ for all bounded and continuous functions g.

Proof. Let $f(t) = E[e^{it\xi_n}]$. Then

$$f^{(m)}(t) = E[i^m \xi_n^m e^{it\xi_n}], \tag{5.3}$$

and

$$\begin{aligned} E[e^{it\sum_{n=1}^N \xi_n/\sqrt{N}}] &= f\left(\frac{t}{\sqrt{N}}\right)^N \\ &= \left(f(0) + \frac{t}{\sqrt{N}} f'(0) + \frac{1}{2}\frac{t^2}{N} f''(0) + o\left(\frac{t^2}{N}\right)\right)^N. \end{aligned}$$

The representation (5.3) implies

$$\begin{aligned} f(0) &= E[1] = 1, \\ f'(0) &= iE[\xi_n] = 0, \\ f''(0) &= -E[\xi_n^2] = -1. \end{aligned}$$

Therefore

$$\begin{aligned} E[e^{it\sum_{n=1}^N \xi_n/\sqrt{N}}] &= \left(1 - \frac{t^2}{2N} + o\left(\frac{t^2}{N}\right)\right)^N \\ &\to e^{-t^2/2}, \quad \text{as } N \to \infty \\ &= \int_{\mathbb{R}} \frac{e^{itx} e^{-x^2/2}}{\sqrt{2\pi}} dx, \end{aligned} \tag{5.4}$$

and we conclude that the Fourier transform (i.e. the characteristic function) of $\sum_{n=1}^N \xi_n/\sqrt{N}$ converges to the right limit of Fourier transform of the standard normal distribution. It is a fact, cf. [D], that convergence of the Fourier transform together with continuity of the limit Fourier transform at 0 implies weak convergence, so that $\sum_{n=1}^N \xi_n/\sqrt{N} \rightharpoonup \nu$, where ν is $N(0,1)$. The exercise below verifies this last conclusion, without reference to other results. □

Exercise 5.3 Show that (5.4) implies

$$E[g(\sum_{n=1}^{N} \xi_n/\sqrt{N})] \to E[g(\nu)] \tag{5.5}$$

for all bounded continuous functions g. Hint: study first smooth and quickly decaying functions g_s, satisying $g_s(x) = \int_{-\infty}^{\infty} e^{-itx}\hat{g}_s(t)dt/(2\pi)$ with the Fourier transform $\hat{g}_s$ of g_s satisfying $\hat{g}_s \in L^1(\mathbb{R})$; show that (5.4) implies

$$E[g_s(\sum_{n=1}^{N} \xi_n/\sqrt{N})] \to E[g_s(\nu)];$$

then use Chebychevs inequality to verify that no mass of $\sum_{n=1}^{N} \xi_n/\sqrt{N}$ escapes to infinity; finally, let $\chi(x)$ be a smooth cut-off function which is one for $|x| \le N$ and zero for $|x| > 2N$ and split the general bounded continuous function g into $g = g_s + g(1-\chi) + (g\chi - g_s)$, where g_s is an arbitrary close approximation to $g\chi$; use the conclusions above to prove (5.5).

Example 5.4 What is the error of $I_N - I$ in Example 5.1?

Solution. Let the error ϵ_N be defined by

$$\begin{aligned}
\epsilon_N &= \sum_{n=1}^{N} \frac{f(x_n)}{N} - \int_{[0,1]^d} f(x)dx \\
&= \sum_{n=1}^{N} \frac{f(x_n) - E[f(x)]}{N}.
\end{aligned}$$

By the Central Limit Theorem, $\sqrt{N}\epsilon_N \rightharpoonup \sigma\nu$, where ν is $N(0,1)$ and

$$\begin{aligned}
\sigma^2 &= \int_{[0,1]^d} f^2(x)dx - \left(\int_{[0,1]^d} f(x)dx\right)^2 \\
&= \int_{[0,1]^d} \left(f(x) - \int_{[0,1]^d} f(x)dx\right)^2 dx.
\end{aligned}$$

In practice, σ^2 is approximated by

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{n=1}^{N}\left(f(x_n) - \sum_{m=1}^{N}\frac{f(x_m)}{N}\right)^2.$$

$\square$

One can generate approximate random numbers, so called pseudo random numbers, by for example the method

$$\xi_{i+1} \equiv a\xi_i + b \mod n$$

where a and n are relative prime and the initial ξ_0 is called the seed, which determines all other ξ_i. For example the combinations $n = 2^{31},\ a = 2^{16} + 3$ and $b = 0$, or $n = 2^{31} - 1,\ a = 7^5$ and $b = 0$ are used in practise. In Monte Carlo computations, we use the pseudo random numbers $\{x_i\}_{i=1}^N$, where $x_i = \frac{\xi_i}{n} \in [0,1]$, which for $N \ll 2^{31}$ behave approximately as independent uniformly distributed variables.

Theorem 5.5 *The following Box-Müller method generates two independent normal random variables x_1 and x_2 from two independent uniformly distributed variables y_1 and y_2*

$$\begin{aligned} x_1 &= \sqrt{-2\log(y_2)}\ \cos(2\pi y_1) \\ x_2 &= \sqrt{-2\log(y_2)}\ \sin(2\pi y_1). \end{aligned}$$

Sketch of the Idea. The variables x and y are independent standard normal variables if and only if their joint density function is $e^{-(x^2+y^2)/2}/2\pi$. We have

$$e^{-(x^2+y^2)/2}dxdy = re^{-r^2/2}drd\theta = d(e^{-r^2/2})d\theta$$

using $x = rcos\theta,\ y = rsin\theta$ and $0 \le \theta < 2\pi,\quad 0 \le r < \infty$. The random variables θ and r can be sampled by taking θ to be uniformly distributed in the interval $[0, 2\pi)$ and $e^{-r^2/2}$ to be uniformly distributed in $(0, 1]$, i.e. $\theta = 2\pi y_1$, and $r = \sqrt{-2log(y_2)}$. $\square$

Example 5.6 Consider the stochastic differential equation $dS = rSdt + \sigma SdW$, in the risk neutral formulation where r is the riskless rate of return and σ is the volatility. Then

$$S_T = S_0\ e^{rT - \frac{\sigma^2}{2}T + \sigma\sqrt{T}\nu}$$