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DN2253, Numerical Algebra: Large matrices

Reading instructions and Review Questions:

The following instructions and questions are intended to be a help when reading the course and preparing for exam. Among the questions, some may be answered just by reading the text, while some need some hand computation. Even if MATLAB is helpful when preparing the course, only hand calculation is needed for these questions.

Instructions refer to sections in the Demmel text book (D), the Eigentemplate book (E) or my lecture notes, Topics in Numerical linear Algebra, (R). The questions are numbered in one sequence.

D 2.2, R 1.4.2: Linear Systems, perturbation theory. Gives bounds on how perturbations in data give perturbations in results. How wanted quantities, perturbations of results, can be found from computable quantities, residuals.

Q 1. Derive an expression for the perturbation of the solution x of a linear system $Ax = b$ when the right hand side b is perturbed.

Q 2. For which perturbation δb does δx get maximal norm, giving equality in the perturbation bound? Use the Euclidean vector norm and singular value decomposition!

Q 3. Derive an expression for the perturbation of the solution x of a linear system $Ax = b$ when the matrix A is perturbed.

Q 4. Assume that we have computed an approximate solution $\hat{x}$ with residual vector $r = b - A\hat{x}$. Show that you can find a perturbation δA, such that the computed solution $\hat{x}$ is the exact solution of the perturbed system

$$(A + \delta A)\hat{x} = b\,.$$

Q 5. Show that the perturbation δA of previous question can be chosen to have $\|\delta A\|_2 = \frac{\|r\|_2}{\|\hat{x}\|_2}$ when we use the Euclidean norm $\| \ldots \|_2$!

Q 6. How large is the smallest perturbation that makes the matrix A singular? Use the singular value decomposition to find one such perturbation!

Q 7. What is the componentwise relative condition number $\kappa_{CR}(A)$.

Q 8. Mention a class of matrices for which $\kappa_{CR}(A)$ is significantly smaller than the standard condition number.

D 2.4, R 1.4.3: Rounding errors in Gaussian elimination. This backward error analysis is important for many numerical algorithms. The same techniques are used for many other algorithms than Gaussian elimination. It is important to follow which intermediate quantities that may cause ill behavior of an algorithm.

Q 9. Give a bound on the elements of the perturbation E caused by rounding errors during Gaussian elimination $A + E = LU$, where L and U are the factors computed with floating point arithmetic.

Q 10. What is the maximal growth factor during Gaussian elimination with partial row pivoting?

D 5.2: Perturbation theory for eigenvalues of symmetric matrices. Real symmetric, or more properly Hermitian, matrices have a nearly perfectly conditioned eigenproblem. They are also of great practical use, so it is meaningful to study them separately.

Q 11. Let A be a real symmetric matrix and x an arbitrary nonzero vector. Show that the Rayleigh quotient $\rho(x, A) = \frac{x^T Ax}{x^T x}$ is a good approximation to an eigenvalue $\lambda(A)$ of A. In what sense?

Q 12. Show that for any choice of vector x and scalar β, there is an eigenvalue α_i of A such that $|\alpha_i - \beta| \leq \|Ax - \beta x\|_2$!

Q 13. Show that this distance is minimized for $\beta = \rho(x, A)$!

Q 14. Show that for a real symmetric matrix $\min \lambda(A) \leq \rho(x, A) \leq \max \lambda(A)$!

Q 15. Formulate the Courant Fischer minimax theorem for real symmetric matrices.

Q 16. Show that a perturbation E to a real symmetric matrix moves the eigenvalues at most $\|E\|_2$ away. (Weyl's theorem)

Q 17. Show that, if $B = A + M$ where M is positive definite, then $\lambda_i(B) \geq \lambda_i(A)$.

Q 18. Show that $\lambda_{\min}(Q^T AQ) \geq \lambda_{\min}(A)$ where Q is an $n \times k$ matrix with orthonormal columns.

D 6.6.1, R4.2: Krylov subspaces, Arnoldi algorithm.

Q 19. What is a Krylov subspace?

Q 20. The Arnoldi algorithm has a basic recursion

$$AQ_k = Q_k H_{k,k} + R$$

discuss properties of the basis Q_k, the reduced matrix $H_{k,k}$ and the residual R!

Q 21. Show that if θ is an eigenvalue and s is an eigenvector of $H_{k,k}$, $H_{k,k}s = s\theta$, then $y = Q_k s$ gives an approximate eigenvector of A. What is the Euclidean norm and direction of its residual $r = Ay - y\theta$?

Q 22. Describe the Krylov algorithm.

Q 23. Show how the Arnoldi algorithm can be derived from the Krylov algorithm by means of a QR factorization.

Q 24. Show that, when Arnoldi is applied to a real symmetric matrix, one gets the Lanczos algorithm.

Q 25. Show that the approximate eigenvectors and their residuals can be expressed as polynomials of A operating on the startiing vector!

Q 26. What are the minimizing properties of the residual polynomials in the real symmetric case?

E 4, E 7, D 7: Lanczos for eigenvalues.

Q 27. When do the basis vectors in the Lanczos algorithm lose orthogonality?

Q 28. What happens when Lanczos is applied to a matrix with multiple eigenvalues?

Q 29. Describe full and selective reorthogonalization. In which cases should one choose one or the other of these?

Q 30. Describe shift invert spectral transformation. In which cases is it applied?

Q 31. Describe implicit restart for the Arnoldi algorithm.

Q 32. How do you get approximate eigenvalues and eigenvectors with the nonsymmetric Lanczos algorithm?

Q 33. What is the optimality properties of these approximations?

Q 34. What is break down in the nonsymmetric Lanczos algorithm?

Q 35. When is nonsymmetric Lanczos to be preferred to Arnoldi?

R 5, D 6.6.2-6: Linear systems, iterative algorithms. This chapter in D starts with a discussion on matrices form simple finite difference approximations of the Poisson's equation over rectangular regions. Table 6.1 on p 277 gives an overview. The basic iterative methods in 6.5 are mainly of historic interest. We are interested in the Krylov subspace methods of 6.6, also described in R 5. For matrices coming from discretizations of partial differential equations, methods that make use of the properties of the underlying problem are the most effective, see the simple case of multigrid in D 6.9 and the short discussion of domain decomposition in D 6.10.

Q 36. Show how the Arnoldi algorithm can be used to get approximate solutions to a linear system of equations.

Q 37. Show how the GMRES (Generalized Minimal Residual) algorithm is one variant of the answer of previous question!

Q 38. Show how the Lanczos algorithm is used to solve a system with a symmetric matrix A!

Q 39. Show that the conjugate algorithm can be derived from the Lanczos algorithm when the matrix A is positive definite!

Q 40. What is meant with that two vectors are A conjugate?

Q 41. Show that the search directions p_k in the conjugate gradient algorithm are A conjugate!

Q 42. Show that the residuals r_k in the conjugate gradient algorithm are orthogonal to each other!

Q 43. Show that the residuals r_k in conjugate gradient can be expressed as a polynomial $q_k(A)b$! What is its minimizing properties?

Q 44. Show that if A has just p distinct eigenvalues, cg will converge after $k = p$ iterations!

Q 45. What is a preconditioning of a matrix?

Q 46. Describe incomplete Cholesky preconditioned conjugate gradient, ICCG. What is meant by ICCG(0) and ICCG(1)?

Q 47. Show that the nonsymmetric (two sided) Lanczos algorithm applied to the matrix A of a linear system $Ax = b$ leads to the Quasi Minimal Residual (QMR) algorithm!

Q 48. Give some advantages and disadvantages of QMR compared to GMRES!

E 6: Computing the Singular Value Decomposition.

Q 49. Given a $m \times n$ matrix A. Define the $(m + n) \times (m + n)$ Hermitian matrix

$$H(A) \equiv \begin{bmatrix} 0 & A \\ A^H & 0 \end{bmatrix}$$

Show that the eigenvalues of H, $\lambda(H) = \pm\sigma(A)$, plus and minus the singular values of A and $|m - n|$ zero eigenvalues! Which are the eigenvectors of H?

Q 50. Show that the Lanczos algorithm applied to $H(A)$, starting at the vector $x_1 = (0, v_1^T)^T$ of order $m+n$, gives one Golub Kahan bidiagonalization of A!

Q 51. Show that starting at $x_1 = (u_1^T, 0)^T$, we get another bidiagonalization.

Q 52. Describe the LSQR iteration for an overdetermined linear system

$$min_x \|Ax - b\|_2$$

Good Luck!