1 About Riemann invariants

Starting from

$$\mathbf{q}_t + \mathbf{f}(\mathbf{q})_x = \mathbf{q}_t + \mathbf{A}(\mathbf{q})\mathbf{q}_x = 0, \mathbf{A} = \frac{\partial \mathbf{f}}{\partial \mathbf{q}}, \mathbf{A}(\mathbf{q}) = \mathbf{R}(\mathbf{q}) diag(\lambda_i(\mathbf{q}))\mathbf{R}^{-1}$$

where the columns of **R** are the right eigenvectors of **A**, we obtain

$$\sum_k l_{ik}(\mathbf{q}) q_{k,t} + \lambda_i(\mathbf{q}) \sum_k l_{ik}(\mathbf{q}) q_{k,x} = 0, i = 1,2,...,s; \quad (\mathbf{R}^{-1})_{ij} = l_{ij}$$

In some cases, these equations admit integration. IF there is an integrating factor μ such that

$$\mu_i(\mathbf{q}) \sum_k l_{ik} dq_k = dR_i(\mathbf{q})$$

THEN R_i is called a Riemann invariant, and it is constant along the characteristic curve

$$\frac{dx}{dt} = \lambda_i(\mathbf{q})$$

Note that these characteristic curves are not straight, because the speeds depend on all the q_k, and only R_i is constant along the curve.

There remains the problem of finding the integrating factor. But one can state that Riemann invariants exist for the case s = 1 and 2 (one or two state variables). For scalar equations, the Riemann invariant is q itself.

For s = 2, we need to integrate $a(u,v)du + b(u,v)dv = 0, \mathbf{q} = (u,v)^T$

The scalar ODE $\frac{dv}{du} = -\frac{a(u,v)}{b(u,v)}$ admits a solution, at least locally, say $v=H(u)$. Then, $R(u,v) = v - H(u)$, and hence any function $f(R)$, is a Riemann invariant. For s = 3, it may or may not be possible to integrate the equations.

Example 1

The shallow water equations

$$h_t + uh_x + hu_x = 0$$
$$u_t + uu_x + gh_x = 0,$$

have

$$\lambda^+ = u + c, \lambda^- = u - c, c = \sqrt{gh}, \mathbf{r}^+ = \begin{pmatrix} h \\ c \end{pmatrix}, \mathbf{r}^- = \begin{pmatrix} h \\ -c \end{pmatrix}, \mathbf{R}^{-1} = \frac{1}{hc}\begin{pmatrix} c & h \\ c & -h \end{pmatrix}$$

Now,

$$cdh + hdu = \sqrt{gh}dh + hdu = h\left(\sqrt{\frac{g}{h}}dh + du\right) = hd(2\sqrt{gh} + u)$$

so the two Riemann invariants are $R^{\pm} = 2\sqrt{gh} \pm u$. Isothermal and isentropic gas dynamics equations admit very similar integrations, and are recommended as exercises.

Example 2

The Euler equations of perfect gas dynamics also allow closed form Riemann invariants. See L. p 293 ff. The Euler equations in primitive variables (ρ,u,p) and quasilinear form are:

$$\begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_t + \begin{pmatrix} u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \gamma p & u \end{pmatrix} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix}_x = 0$$

has characteristic eigenvalues $\lambda_1 = u, \lambda_2 = u - c, \lambda_3 = u + c, c^2 = \frac{\gamma p}{\rho}$ (different numbering from L.) with eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -\rho/c \\ 1 \\ -\rho c \end{pmatrix}, \begin{pmatrix} \rho/c \\ 1 \\ \rho c \end{pmatrix}$$

and one finds the inverse of the R-matrix to be

$$\begin{pmatrix} 1 & 0 & -1/c^2 \\ 0 & 1/2 & -1/(2\rho c) \\ 0 & 1/2 & 1/(2\rho c) \end{pmatrix}$$

So we try to integrate:

$$d\rho - dp/c^2, \quad du - dp/(\rho c), \quad du + dp/(\rho c)$$

The first differential involves only two variables so is integrable; it has integrating factor $1/\rho$:

$$d\rho \cdot \frac{1}{\rho} - dp \cdot \frac{\rho}{\gamma p} \cdot \frac{1}{\rho} = d \ln \rho - d \ln(p^{\frac{1}{\gamma}}) = \frac{1}{\gamma} d\left(\frac{\rho^\gamma}{p}\right)$$

which shows that the entropy, say $S = p/\rho^\gamma$, is constant along particle paths through smooth solutions.

The two other differentials involve three variables and need help to be integrated. Using constant entropy, one can express ρ in S and p: $\rho = (p/S)^{1/\gamma}$ and reduce the differential to *two* variables, u and p, and after integration, replace S by p/ρ^γ again. The final result is

$$R^{\pm} = u \pm \frac{2c}{\gamma - 1}$$