KTH Computer Science
and Communication

Lecture Notes 2

Heat Equation

1 Derivation

Denote the temperature $T(t, x)$ [K], with $x \in \mathbb{R}^3$, and the internal energy per unit mass $H(T)$ [J]. For a solid (or liquid) a small change of temperature leads to a small change in internal energy[1],

$$dH = C(T)dT.$$

The coefficient $C(T)$ is called the specific heat, [J/(K kg)]. We furthermore let the density be ρ [kg/m^3] and the heat flux $\vec{F}$, [W/m^2]. We also assume that there is a heat source $S(t, x)$ [W/m^3] (from combustion, ohmic electric heating, ...)

In this setting the thermal energy ρH is conserved. As before, this gives the conservation law in integral form,

$$\frac{d}{dt}\int_V \rho H(T)dV + \int_S \vec{F}\cdot\vec{n}dS = \int_V SdV,$$

for any volume V with surface S. We also get the conservation law in differential form,

$$\frac{\partial}{\partial t}(\rho H(T)) + \nabla\cdot\vec{F} = S.$$

We suppose ρ is independent of time, such that

$$\frac{\partial}{\partial t}(\rho H(T)) = \rho\frac{\partial H(T)}{\partial t} = \rho C(T)\frac{\partial T}{\partial t}.$$

To complete the derivation we use Fourier's law, which states that the heat flux $\vec{F}$ is in the direction of the negative temperature gradient: $\vec{F} = -k\nabla T$, where k is the heat conductivity, units [W/(mK)]. The final form of the heat equation is

$$\rho C(T)\frac{\partial T}{\partial t} - \nabla\cdot(k\nabla T) = S.$$

This is the correct form also when the data ρ, C and k vary with position and with T. If the coefficients are constant, it reduces to

$$\frac{\partial T}{\partial t} - \alpha\Delta T = \frac{S}{\rho C}, \qquad \alpha = \frac{k}{\rho C},$$

where α is called the thermal diffusivity with units [m^2/s].

When the heat equation models heat conduction inside a domain Ω, natural boundary conditions are

$$k\nabla T\cdot\vec{n} = h(T_e - T), \qquad x \in \partial\Omega,$$

[1] At phase changes, however, $H(T)$ will have a jump corresponding to the latent heat.

which represents conductive cooling of the domain. Here h [W/(m^{2}K)] is the heat transfer coefficient and T_e is the surrounding temperature.

In the following we will often formulate the models in non-dimensional quantities. Here is an example. Suppose ρC is constant and that k_0 is a typical value of $k(x)$. Choose temperature scale T_s, length-scale L, and time scale $t_s = \rho C L^2/k_0$. Then introduce the scaled non-dimensional variables $q = (T - T_e)/T_s$, $y = x/L$ and $\tau = t/t_s$. The heat equation for $q = q(\tau, y)$ becomes,

$$\frac{\partial q}{\partial \tau} - \nabla \cdot (\tilde{k}(y)\nabla q) = \tilde{S}(\tau, y), \qquad \tilde{k}(y) = k(yL)/k_0, \quad \tilde{S}(\tau, y) = S(\tau t_s, yL)\frac{L^2}{T_s k_0},$$

with boundary conditions[2]

$$\frac{\partial q}{\partial n} + bq = 0, \qquad b = \frac{hL}{k_0}.$$

The non-dimensional coefficient $b > 0$ is called the Biot number. It represents the ratio of thermal resistance inside the domain (L/k) and at the boundary ($1/h$). For very small Biot numbers the boundary condition can be replaced by the Neumann condition $\partial q/\partial n = 0$. Large Biot numbers, on the other hand, leads to the Dirichlet condition $q = 0$.

The final initial boundary value problem that we consider is

$$\begin{aligned} u_t - \nabla \cdot (k(x)\nabla u) &= S(t, x), & x &\in \Omega, \quad t > 0, \\ u(0, x) &= f(x), & x &\in \Omega, \\ \frac{\partial u}{\partial n} + b(x)u &= 0, & x &\in \partial\Omega, \quad t \geq 0, \end{aligned} \tag{1}$$

where Ω is an open domain with boundary $\partial\Omega$, $k(x) > 0$ and $b(x) \geq 0$.

2 Well-posedness

In order to check well-posedness we need to show existence of a solution and an energy estimate.

2.1 Existence

We will do this in a simplified setting. We consider the 2D case and take $\Omega = (0, 2\pi)^2$, $k(x, y) \equiv 1$ and $b(x, y) = S(t, x, y) \equiv 0$, so that

$$\begin{aligned} &u_t - \Delta u = 0, && 0 < x < 2\pi, \quad 0 < y < 2\pi, \quad t > 0, \\ &u(0, x) = f(x), && 0 < x < 2\pi, \quad 0 < y < 2\pi, \\ &u_x(t, 0, y) = u_x(t, 2\pi, y) = u_y(t, x, 0) = u_y(t, x, 2\pi) = 0, && 0 < x < 2\pi, \quad 0 < y < 2\pi, \quad t \geq 0. \end{aligned}$$

Then we can explicitly construct a solution via Fourier analysis. Write the solution in terms of a cosine series,

$$u(t, x, y) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \hat{u}_{k\ell}(t)\cos(kx)\cos(\ell y).$$

This satisfies the boundary conditions, and to satisfy the initial condition we choose $\hat{u}_{k\ell}(0) = \hat{f}_{k\ell}$, the corresponding coefficients of the cosine series for f. Inserting the series in the equation gives

$$u_t = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{d\hat{u}_{k\ell}(t)}{dt}\cos(kx)\cos(\ell y) = u_{xx} + u_{yy} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} -(k^2 + \ell^2)\hat{u}_{k\ell}(t)\cos(kx)\cos(\ell y),$$

[2] $\frac{\partial q}{\partial n} := \nabla q \cdot \vec{n}$

so that

$$\frac{d\hat{u}_{k\ell}(t)}{dt} = -(k^2+\ell^2)\hat{u}_{k\ell}(t),$$

with solution $\hat{u}_{k\ell}(t) = \hat{u}_{k\ell}(0)\exp(-(k^2+\ell^2)t)$. Finally,

$$u(t,x,y) = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \hat{f}_{k\ell}\cos(kx)\cos(\ell y)e^{-(k^2+\ell^2)t}.$$

This shows existence of a solution for this simple case. Note:

- High frequencies in the initial data (large k, ℓ) are damped fast. This means that rough initial data (= many high frequencies) is rapidly smoothed, or "smeared".
- For the backward heat equation we would have $e^{+(k^2+\ell^2)t}$ instead of $e^{-(k^2+\ell^2)t}$, which instead amplifies high frequencies, more the higher they are. Small perturbations will then quickly destroy the solution.
- For the general setting proving existence is more complicated and beyond the scope of the course. One standard way is to design a numerical method for the problem and show 1) that it converges (by compactness or completeness) and 2) that the limit solution obtained indeed satisfies the PDE.

2.2 Energy estimate

We consider the full case (1) and make the estimate in L^2-norm,

$$||u(t,\cdot)||^2 = \int_\Omega u(t,x)^2 dx.$$

Then

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2 &= \int_\Omega u(t,x)u_t(t,x)dx = \{\text{use the PDE (1)}\} = \\
&= \int_\Omega u\nabla\cdot(k(x)\nabla u)dx + \int_\Omega uSdx = \{\text{integration by parts}\} = \\
&= \int_\Omega -k(x)|\nabla u|^2dx + \int_{\partial\Omega} k(x)u\frac{\partial u}{\partial n}dx + \int_\Omega uSdx = \{\text{use bc in (1)}\} = \\
&= -\int_\Omega k(x)|\nabla u|^2dx - \int_{\partial\Omega} b(x)k(x)u^2dx + \int_\Omega uSdx \leq \quad \{b(x), k(x) \geq 0\} \\
&\leq \int_\Omega uSdx.
\end{aligned}$$

We finally use the Cauchy-Schwarz inequality,

$$\int_\Omega f(x)g(x)dx \leq \left(\int f(x)^2dx\right)^{1/2}\left(\int g(x)^2dx\right)^{1/2} = ||f||\cdot||g||,$$

and conclude that

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2 \leq ||u(t,\cdot)||\cdot||S(t,\cdot)||.$$

Assuming $||u(t,\cdot)|| \neq 0$, we can divide both sides by $||u(t,\cdot)||$ so that

$$\frac{d}{dt}||u(t,\cdot)|| \leq ||S(t,\cdot)||.$$

Upon integrating this in time we get

$$||u(t,\cdot)|| - ||u(0,\cdot)|| \le \int_0^t ||S(\tau,\cdot)||d\tau \le \int_0^T ||S(\tau,\cdot)||d\tau,$$

for all $0 \le t \le T$. Hence,

$$||u(t,\cdot)|| \le ||f|| + \int_0^T ||S(\tau,\cdot)||d\tau, \qquad 0 \le t \le T,$$

which is the desired energy estimate (with $C = 1$).

3 Properties

Here we discuss some properties of the heat equation.

Smoothing

As we saw above, high frequencies in initial data are damped quickly and the solution is therefore smooth for $t > 0$. In fact, even for initial data in L^2, which can be arbitrarily rough, the mapping $x \mapsto u(t,x)$ (with fixed t) is *analytic* for all $t > 0$, i.e. very smooth. (Assuming for instance that $k \equiv 1$ and $S \equiv 0$.)

Decay of L^2-norm

While proving the energy estimate above, we obtained as an intermediate result that

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2 = -\int_\Omega k(x)|\nabla u|^2 dx - \int_{\partial\Omega} b(x)k(x)u^2 dx \le -\int_\Omega k(x)|\nabla u|^2 dx,$$

when $S \equiv 0$. We simply estimated this by ≤ 0, but typically the right hand side remains strictly smaller than zero and there is a monotone decrease in time of the L^2-norm $||u(t,\cdot)||$. When we have Dirichlet conditions, $u = 0$ on $\partial\Omega$, this follows directly from one version of the Poincaré inequality, which says that, for any sufficiently smooth function $v(x)$ that is zero on the boundary, there is a number C such that

$$||v|| \le C||\nabla v||, \tag{2}$$

provided Ω is smooth enough, open and connected. The number C only depends on the shape of Ω (not on v!). Let $k_m := \inf_{x\in\Omega} k(x) > 0$. Using (2) we get

$$\frac{1}{2}\frac{d}{dt}||u(t,\cdot)||^2 \le -k_m \int_\Omega |\nabla u|^2 dx \le -Ck_m||u(t,\cdot)||^2.$$

Let $z = \exp(2Ck_m t)||u(t,\cdot)||^2$. Then

$$\frac{dz}{dt} = 2Ck_m z + \exp(2Ck_m t)\frac{d}{dt}||u(t,\cdot)||^2 \le 2Ck_m z + -2Ck_m \exp(2Ck_m t)||u(t,\cdot)||^2 = 0.$$

Hence, $z(t) \le z(0)$, or

$$||u(t,\cdot)|| \le \exp(-Ck_m t)||u(0,\cdot)||.$$

The decay of the L^2-norm is thus exponential.

Maximum principle

The maximum principle for the heat equation says that when $S = 0$ the maximum value of $u(t, x)$ in $[0, T] \times \bar{\Omega}$ is either obtained on the boundary $x \in \partial\Omega$ or for the initial data at $t = 0$. Note that this is true also when k depends on x and for any boundary condition.

Moreover, there is no amplification of local spatial extrema: local spatial maximum (minimum) of u in Ω cannot increase (decrease). Indeed, suppose u has a local maximum at x^* at time t. Then $\nabla u(t, x^*) = 0$ and $D^2 u(t, x^*)$ is semi-negative definite, in particular $\Delta u(t, x^*) \leq 0$. It follows that

$$\frac{\partial u(t,x^*)}{\partial t} = \nabla \cdot (k(x^*)\nabla u(t,x^*)) = \nabla k(x^*) \cdot \nabla u(t,x^*) + k(x^*)\Delta u(t,x^*) = k(x^*)\Delta u(t,x^*) \leq 0,$$

since $k(x) > 0$. The local maximum will thus not increase.

In one dimension the *total variation* of the solution is non-increaseing. The total variation for a function $v(x)$ on the domain $[a, b]$ is defined as

$$TV(v) := \int_a^b |v_x| dx = \sum_{j=0}^{n-1} |v(x_{j+1}) - v(x_j)|,$$

where $x_1, \ldots, x_{n-1}$ are the local extrema in (a, b) and $x_0 = a$, $x_n = b$. This follows essentially from the statement above that $u(t, x_j)$ does not increase (decrease) if it is a local maximum (minimum), i.e. $|u(t, x_{j+1}) - u(t, x_j)|$ decreases.

Conservation

The integral of the solution u over Ω is constant in time if $S \equiv 0$ and the boundary conditions are Neumann conditions, $\partial u/\partial n = 0$.

$$\frac{d}{dt}\int_\Omega u(t,x)dx = \int_\Omega u_t(t,x)dx = \int_\Omega \nabla \cdot (k(x)\nabla u)dx = \int_{\partial\Omega} k(x)\frac{\partial u}{\partial n}dx = 0.$$

This comes as no surprise – it is the basis on which the PDE was derived. In fact, it holds for any conservation law with "no flux" boundary condition, $\vec{F} \cdot \vec{n} = 0$,

$$u_t + \nabla \cdot \vec{F} = 0 \quad \Rightarrow \quad \frac{d}{dt}\int_\Omega u(t,x)dx = -\int_\Omega \nabla \cdot \vec{F} dx = -\int_{\partial\Omega} \vec{F} \cdot \vec{n} dx = 0.$$

When $S \neq 0$, we get instead that

$$u_t + \nabla \cdot \vec{F} = S \quad \Rightarrow \quad \frac{d}{dt}\int_\Omega u(t,x)dx = \int_\Omega S(t,x)dx$$

$$\Rightarrow \quad \int_\Omega u(t,x)dx = \int_\Omega u(0,x)dx + \int_0^t \int_\Omega S(\tau,x)dxd\tau.$$

Inifinite speed of propagation

A spatially localized change in initial data will in general change the solution for all x immediately, i.e. for any $t > 0$. For example, if

$$\begin{aligned} u_t - \nabla \cdot (k(x)\nabla u) &= S(t,x), & x \in \mathbb{R}^n, \quad t > 0, \\ u(0,x) &= f(x), & x \in \mathbb{R}^n, \end{aligned}$$

and

$$
\begin{aligned}
v_t - \nabla \cdot (k(x)\nabla v) &= S(t,x), && x \in \mathbb{R}^n, \quad t > 0, \\
v(0,x) &= f(x) + \delta(x), && x \in \mathbb{R}^n,
\end{aligned}
$$

where $\delta(x)$ is zero outside a small ball $|x - x_0| \leq \varepsilon$, then in general $u(t,x) \neq v(t,x)$ for all x and $t > 0$. Hence, the perturbation $\delta(x)$ travels at "infinite speed" and affects the solution everywhere, in infinitesimal time. This is in sharp contrast to hyperbolic problems, for which a perturbation has "finite speed" and the two solutions u and v would be identical outside the ball $|x - x_0| < \varepsilon + Ct$, for some $C > 0$.

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