Examination paper Numerical Solution of DE, 2D1255/DN2255

09-14, Sep 17, 2011

Closed books examination. Read all the questions before starting work. Check carefully that your initially derived equations are correct. ***Ask if you are uncertain*** !.
Answers MUST be motivated. You can judge the level of details required in the answers from the number of points. Paginate and write your name on EVERY page handed in.
A total of $N/2$ out of max N (= 50) points guarantees a "pass". The results will be e-mailed to participants by Sep 14, 2011. Papers are kept at the CSC Student Office for a year and then destroyed. Complaints to J.Oppelstrup by Dec 1, 2011, after which the results are irrevocable.

P1. (9) Consider the Riemann problem for a discontinuity with left state q_L and right state q_R at (0,0) for the conservation law

$$q_t + f(q)_x = 0, f(q) = q(1-q)$$

(a) (3) Determine for which part of the (q_L,q_R)-plane the solution is a *shock*. Hint: Lax' condition of shock collision. What kind of solution for the rest? You need not prove your answer, which follows from the fact that $f(.)$ is convex.

(b) (3) For the shock solution, determine the shock speed and the "Godunov state" $q^{\vee\!/}$ (Leveque's notation), i.e. $q(x=0,t=0+)$.

(c) (3) For the "other kind of solution", find $q(x,t)$ as a similarity solution

$$q(x,t) = v(\xi), \xi = x/t$$

and from that find $q^{\vee\!/}$ for these (q_L,q_R)

P2. (7) Consider a Cauchy-problem for the first order system (a,b,c real)

$$\mathbf{q}_t + \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mathbf{q}_x = 0$$

a) (2) What is the condition on a,b,c that the system be hyperbolic? symmetric hyperbolic?
The Lax-Friedrichs scheme for a constant coefficient hyperbolic first order system $\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = 0$ is

$$\mathbf{Q}_j^{n+1} = \frac{1}{2}(\mathbf{Q}_{j+1}^n + \mathbf{Q}_{j-1}^n) - \frac{\Delta t}{2\Delta x}\mathbf{A}(\mathbf{Q}_{j+1}^n - \mathbf{Q}_{j-1}^n)$$

b) (2) Compute the von Neumann magnification matrix **G** for an ansatz

$$\mathbf{Q}_j^n = \mathbf{U}^n e^{ijk\Delta x}$$

c) (3) For $a = 6$, $b = c = 4$, what are its eigenvalues? For which values of Δt and Δx is the method stable?

P3. (10)

a) (2) Explain the concepts mathematical and numerical domain of dependence for a hyperbolic system, with characteristic speeds $a1 > 0$ and $a2 < 0$.
Sketch the grid in (x,t)-space.

b) (3) Explain, and ***again make a proper sketch***, the CFL non-convergence criterion for a three-point consistent scheme for the equation in P3 a)

c) (2) Explain what is meant by the total variation of a function f on an interval $[0,L]$, TV(f)

d) (2) Compute the total variation on [0.5,2.9] of the function

$$f(x) = 2x - \text{int}(x),$$

where int(x) is the largest integer $\leq x$

(cont'd overleaf)

P4. (6) The shallow water Riemann problem

$$h_t + (hu)_x = 0$$
$$(hu)_t + (hu^2 + 1/2gh^2)_x = 0$$
$$h(x,0) = hL, x < 0, hR, x > 0$$
$$u(x,0) = uL, x < 0, uR, x > 0$$

where $g = 9$ m/s^2 is the gravitational acceleration has a left state $hL = 1$, $uL = 4$ for $x < 0$. The right state, for $x > 0$, is $\mathbf{q}R = (hR,uR)$.

a) (3) Sketch the characteristics starting from $x < 0$. Is this a supercritical or sub-critical flow?

b) (3) Find conditions on **q**R (if possible) to make the solution a steady shock.

P5. (8) Consider the system of conservation laws $\mathbf{q}_t + (\mathbf{f}(\mathbf{q}))_x = 0$. A time-stepping scheme in conservation form is

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^{n+1} - \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{j+1/2}^n - \mathbf{F}_{j-1/2}^n\right)$$

The Roe scheme has numerical flux

$$\mathbf{F}_{j-1/2}^n = \frac{1}{2}\left(\mathbf{f}(\mathbf{Q}_{j-1}^n) + \mathbf{f}(\mathbf{Q}_j^n)\right) - \frac{1}{2}\left|\mathbf{A}_{j-1/2}^n\right|\left(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n\right)$$

a) (2) Define the meaning of $\mathbf{A}^+$, $\mathbf{A}^-$, and $|\mathbf{A}|$ as used in the Roe-scheme. **A** is a real square matrix.

The Roe matrix $\mathbf{A}_{j-1/2} = \mathbf{A}_{j-1/2}(\mathbf{Q}_{j-1},\mathbf{Q}_j)$ should satisfy three conditions,

b) (2) i) what should the limit of $\mathbf{A}_{j-1/2}$ be when $\mathbf{Q}_{j-1} \to \mathbf{Q}, \mathbf{Q}_j \to \mathbf{Q}$?

(1) ii) what properties of eigenvectors and eigenvalues?

It should also satisfy the "conservation" requirement

(iii) $\mathbf{A}_{j-1/2} \cdot \left(\mathbf{Q}_j - \mathbf{Q}_{j-1}\right) = \mathbf{f}(\mathbf{Q}_j) - \mathbf{f}(\mathbf{Q}_{j-1})$

c) (3) Show that the update formula can be written (using iii))

$$\mathbf{Q}_j^{n+1} = \mathbf{Q}_j^n - \frac{\Delta t}{\Delta x}\left(\mathbf{A}_{j-1/2}^+(\mathbf{Q}_j^n - \mathbf{Q}_{j-1}^n) + \mathbf{A}_{j+1/2}^-(\mathbf{Q}_{j+1}^n - \mathbf{Q}_j^n)\right)$$

P6. (9) Consider the model below for a counter-flow heat exchanger,

$$\mathbf{q}_t + \mathbf{A}\mathbf{q}_x = \mathbf{B}\mathbf{q}, \mathbf{q} = \begin{pmatrix} u \\ v \end{pmatrix}, \mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & -b \end{pmatrix}, \mathbf{B} = \frac{1}{\tau}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

$a > 0$ and $b > 0$, $a > b$, are the velocities of the two streams, and $\tau > 0$ the small time-scale of heat transfer between the streams. The **Bq** terms express the "reaction" – actually heat transfer between the streams. Discretize in space by a finite volume scheme, cell size Δx. Let $\mathbf{Q}^n$ be the totality of all u and v at time t^n. The time-stepping is done by a Godunov (first order) splitting scheme, time step Δt, with the first order upstream scheme for the convection step, C : $\mathbf{Q}^{n-1}$ -> **Q***, and the implicit Euler scheme for the "reaction" step R: **Q*** -> $\mathbf{Q}^n$:

(a) (3) Define *by formulas and explain* the procedure which computes $\mathbf{Q}^n$ from $\mathbf{Q}^{n-1}$;

Use vonNeumann analysis to show that the overall scheme is stable subject to the CFL condition, in two steps:

(b) (3) C is a contraction in L_2 if the CFL condition is satisfied;

(c) (3) R is a contraction in L_2 for all positive Δt

Hint: A linear mapping **P** = **F**(**Q**) is called "a contraction in L_2" if $\|\mathbf{P}\|_2 \le \|\mathbf{Q}\|_2$ for all **Q**.