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Numerical treatment of Differential Equations II, 2D1255, 2005

Possible exam questions

1. Which, (if any) of the following equations are

$$(a) \text{ well posed?} \quad (b) \text{ hyperbolic?} \quad (c) \text{ parabolic?}$$

$$u_t + u_x + 2u = 0, \tag{1}$$

$$u_t = u_{xx} - u_{yy}, \tag{2}$$

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0, \tag{3}$$

$$u_t = u_{xxx} + u. \tag{4}$$

2. Show that the initial-value problem for $u_t = u_{xx} + u_x + u$ is well posed. Suppose you want to solve the above equations for $0 \le x \le 1$, $0 \le t \le 1$. State suitable boundary conditions. Show that with your boundary conditions solutions are bounded in norm (which?)

$$\|u(\cdot, t)\| \le C\|u(\cdot, 0)\|.$$

Here C must be independent of the initial condition $u(x, 0)$.

3. Consider a general initial value problem for systems of linear PDE with constant coefficients, in one space dimension, $q_t = P(\partial_x)q$. State precise conditions for wellposedness. Give examples of an ill posed and a wellposed problems. Show that the conditions cannot be met and are met, respectively

4. Derive the Rankine-Hugoniot condition for a system of conservation laws $q_t + (f(q))_x = 0$.

5. Consider $q_t + (q^2/2)_x = 0$ together with initial condition $q(x, 0) = 1$ if $x < 0$ $q(x, 0) = a$ if $x > 0$. Solve the initial-value problem for the two cases $a = 2$ and $a = 0$.

6. Consider the system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = 0,$$

in $0 \le x \le 1$. Which of the following boundary conditions yield a well posed problem? Discuss reflections in wellposed cases.

   (a) $u(0, t) = 1$, $v(0, t) = 0$,

   (b) $u(0, t) + v(0, t) = 0$, $u(1, t) = 1$,

   (c) $u(0, t) - v(0, t) = 0$, $u(1, t) + v(1, t) = 1$.

7. Considera scalar conservation law $q_t + (f(q))_x = 0$ approximated by a finite volume method
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n).$$
Here $F_{i+1/2}^n$ is called the numerical flux. Consider the following (or the Lax-Friedrichs or the Lax-Wendroff) scheme
$$Q_i^{n+1} = \frac{1}{2}Q_i^n + \frac{Q_{i+1}^n + Q_{i-1}^n}{4} - \frac{\Delta t}{2\Delta x}(f(Q_{i+1}^n) - f(Q_{i-1}^n)).$$
   (a) What is the numerical flux?
   (b) What is the accuracy?
   (c) Consider the method applied to the advection equation $q_t + uq_x = 0$ with u constant. For which k, h is the method stable?
   (d) Derive the modified equation. Of what type is it? How is the stability limit related to the wellposedness of the modified equation?

8. Consider a scalar advection equation $q_t + (u(x)q)_x = 0$ approximated by a finite volume method
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n).$$
Here $F_{i+1/2}^n$ is called the numerical flux.
   (a) What is the numerical flux for the Upwind method
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(u^+\Delta Q_{i-1/2}^n + u^-\Delta Q_{i-1/2}^n).?$$
   Here $u^+ = \max(u, 0)$, $u^- = \min(u, 0)$ and $\Delta Q_{i-1/2} = Q_i - Q_{i-1}$.
   (b) What is numerical flux of the Lax Wendroff method
$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{2\Delta x}u(Q_{i+1}^n - Q_{i-1}^n) + \frac{1}{2}(\frac{\Delta x}{\Delta t})^2 u^2(Q_{i+1}^n - 2Q_i^n + Q_{i+1}^n)?$$
   (c) Discribe how these fluxes are used to construct a high resolution method.

9. Derive the upwind method for the advection equation $q_t + q_x = 0$ from the following algorithm.
   (1) Reconstruct a piecewiese constant function $\tilde{q}^n(x)$ from the cell avarages Q_i^n.
   (2) Solve
$$q_t + q_x = 0, \quad q(x, t_n) = \tilde{q}^n(x)$$
   until $t_n + \Delta t = t_{n+1}$.
   (3) Avarage over grid cells to obtain new cell avarages.

   Discuss how this approach can be modified to obtain a high resolution method.

10. Consider $q_t + q_x = 0$ discretized. A general one-step method can be written as $Q^{n+1} = N(Q^n)$

(a) What does "N is contractive in the 2-norm" mean?

(b) Define the local truncation order in terms of N.

(c) Show that if a method is contractive in a norm then one can obtain an error bound, valid for all $0 \le t_n \le T$ in terms of $\max_{0\le n\le M} \|\tau^n\|$. Here $T = M\Delta t$ where Δt is the timestep and τ^n is the local truncation error.

(d) State in a precise way what is meant by convergence. Explain when convergence follows from your result.

(e) Consider the Lax-Wendroff (or Lax-Friedrichs or Upwind) method. Show that the corresponding N is a contraction in $\|\cdot\|_2$ if the CFL condition is satisfied.

(f) Consider $q_t + q_x = u$. Modify the requirements on N and derive a corresponding error bound and convergence result.

11. Consider $q_t + (q^2)_x = 0$ discretized by

$$Q_j^{n+1} - Q_j^n = -\frac{k}{h} 2Q_j^n (Q_{j+1}^n - Q_j^n).$$

(a) Is this a conservative discretization? If not, suggest a conservative discretization.

(b) Is this a consistent discretization?

12. Linearize

$$\begin{pmatrix} \rho \\ \rho v \end{pmatrix}_t + \begin{pmatrix} \rho v \\ \rho v^2 + \kappa \rho^\gamma \end{pmatrix}_x = 0,$$

at constants ρ_0 and v_0. Here $\kappa > 0$ and $\gamma \ge 1$ are constants.

(a) Derive the Lax-Wendroff method for the linear system For which values of k and h is the method stable? State in a precise way what is meant by stability.

(b) Consider the linearized problem for $0 \le x \le 1$. Suggest boundary conditions at $x = 0$ and $x = 1$ that yield a mathematically well posed problem. Assume the linearized equations are discretized by the Lax-Wendroff method. Suggest numerical boundary conditions where they are needed.

13. Discretize the linear hyperbolic system $u_t + Au_x = 0$ with forward difference in time and central(or forward or backward) difference in space. Analyze the stability using von Neumann analysis.

14. The scalar equation

$$q_t + q_x = 0, \quad 0 \le x \le 1,$$
$$q(0,t) = g(t), \quad q(x,0) = q_0(x),$$

is approximated by

$$\frac{d}{dt}Q + DQ = -\tau \begin{pmatrix} Q_1 - g(t) \\ 0 \\ \vdots \end{pmatrix}, \quad Q(0) = Q_0.$$

Here $Q(t) = (Q_1(t), \ldots Q_N(t))^T$ where $Q_i(t) \approx q(x_i, t)$ and $x_i = (i-1)\Delta x$, $\Delta x = 1/(N-1)$. Further

$$D = \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & & & \\ -0.5 & 0 & 0.5 & & \\ & \cdot & \cdot & \cdot & \\ & & -0.5 & 0 & 0.5 \\ & & & -1 & 1 \end{pmatrix}$$

(a) Show that D is a Summation-By-Parts operator with respect to the norm $\|v\|_H^2 = \Delta x(0.5v_1^2 + v_2^2 + \ldots + v_{N-1}^2 + 0.5v_N^2)$.

(b) What value should τ have? Why?

15. The scalar equation

$$q_t + q_x = 0, \quad 0 \leq x \leq 1,$$
$$q(0,t) = g(t), \quad q(x,0) = q_0(x),$$

is approximated by

$$\frac{d}{dt}Q + DQ = -\tau \begin{pmatrix} Q_1 - g(t) \\ 0 \\ \vdots \end{pmatrix}, \quad Q(0) = Q_0.$$

Here $Q(t) = (Q_1(t), \ldots Q_N(t))^T$ where $Q_i(t) \approx q(x_i, t)$ and $x_i = (i-1)\Delta x$, $\Delta x = 1/(N-1)$. Further, D is an $N \times n$ matrix.

(a) What is required of the matrix D for the discretization to be called a Summation-By-Parts discretization?

(b) Why is this a desirable property?

16. Consider the heat equation $u_t = \kappa u_{xx}$ with $\kappa > 0$. Analyze the stability of

$$u_j^{n+1} = u_j^n + \frac{\kappa \Delta t}{(\Delta x)^2}(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1}),$$

$$u_j^{n+1} = u_j^n + \frac{\kappa \Delta t}{(\Delta x)^2}(u_{j-1}^{n} - 2u_j^{n} + u_{j+1}^{n}).$$

17. Consider solving the heat equation on a $d-$dimensional unit box in space with Dirichlet boundary conditions, using second order finite difference approximations for the spatial derivatives. Assume $h = 1/N$ is the space step in all space directions, and that accuracy in time requires a time step $k \leq h$. Discuss how the work (flops) to compute until t=1 increases with N in the following two cases. Consider $1, 2$ and 3 space dimensions. (1) An explicit method is used in time. Remember that stability requirements must be satisfied. (2) An implicit method (with good stability properties) is used, and in each time level the system of equations is solved using Gaussian elimination.

18. Explain why explicit time stepping usually is used for hyperbolic problems while implicit time stepping is used for parabolic problems.

19. Consider

$$(1) \quad u_{tt} + u_{xx} = 0,$$

$$(2) \quad u_{tt} - u_{xx} = 0.$$

(a) Introduce $w = (u_x, u_t)^T$ and derive systems of first order equations for w for (1) and (2). Discuss hyperbolicity of the first order systems.

(b) Discuss the possibility of solving (1) and (2) by discretizing the corresponding first order systems with the Lax-Friedrichs method.