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Numerical treatment of Differential Equations II, 2D1255, 2003

Problem set 1

Hand in solutions to the following problems no later then January 29.

You may want to reveiw the material from Edsbergs course 2D1225

1. Let A be a $m \times n$ matrix och x an $n-$vector. What are the dimensions of $A^T A$, Ax, xx^T, $x^T x$?

2. What is meant by eigenvalues and eigenvectors of a square matrix? How many are there? Determine (using MATLAB) the eigenvectors and eigenvalues of
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

3. Let $x_1, ...x_m$ be $n-$ vectors. Explain what it means that the vectors are linearly dependent or linearly independent. How many linearly independent $n-$ vectors can there be?

4. Let
$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad F(x) = \begin{pmatrix} x_1^2 + x_2^2 - 4 \\ x_2 - x_1^2 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$
What is the linear approximation of F at x_0?

5. Determine the solutions of
$$\text{a)} \quad \frac{du}{dt} = \sin(t)u, \quad u(0) = 4,$$
$$\text{b)} \quad \frac{d}{dt}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

6. Solve the following difference equations
$$\text{a)} \quad a_{n+1} = 2a_n - \frac{3}{4}a_{n-1}, \quad a_0 = 1, \quad a_1 = 2.$$
$$\text{b)} \quad a_{n+1} = 2a_n - a_{n-1}, \quad a_0 = 1, \quad a_1 = 2.$$

7. One intends to solve
$$\frac{du}{dt} = \begin{pmatrix} -1 & 6 \\ 0 & -1000 \end{pmatrix} u + \begin{pmatrix} \sin(t) \\ e^{-t} \end{pmatrix}$$
using the classical fourth order Runge-Kutta method. How must the time step be restricted to avoid an unstable computation? **Hint**: The stability region of the method includes the interval $[-2.8, 0]$ on the real axis.

8. Derive the stability region of the forward Euler method.

9. Derive the order of accuracy for the forward Euler method.

10. A circular motion is described by

$$\frac{d}{dt}v = Kv, \quad v(0) = v_0,$$

where

$$v = \begin{pmatrix} x \\ y \end{pmatrix}, \; K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The solution is $x(t) = \cos(t)$, $y(t) = \sin(t)$.

a) Produce an approximate circle by approximately solving the ODE with the Euler, the trapezoidal and the backward Euler methods, respectively.

$$\begin{aligned}
&(1) \quad v^{n+1} - v^n = kKv^n, \\
&(2) \quad v^{n+1} - v^n = \frac{k}{2}K(v^n + v^{n+1}), \\
&(3) \quad v^{n+1} - v^n = kKv^{n+1}.
\end{aligned}$$

Here v^n approximates $v(n \cdot k)$. Plot the solutions in the (x, y) plane.

b) Analysis: In each case introduce a matrix G_i such that $v^{n+1} = G_i v^n$, for $i = 1, 2, 3$. Determine the eigenvalues of G_i and decide from their absolute values whether the picture in the three cases is a spiral towards the origin, a spiral away from the origin or a circle. **Hint**: find the eigenvalues of K and use the fact that if K has eigenvalue λ then $f(K)$ has eigenvalue $f(\lambda)$.

c) Show that in one case all the points are at the same distance from the the origin. **Hint**: $(v^{n+1} + v^n)(v^{n+1} - v^n) = \|v^{n+1}\|^2 - \|v^n\|^2$.