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Numerical treatment of Differential Equations II, 2D1255, 2004

Problem set 4

Hand in a solution no later then March 5.

The relevant background for this problem is 6.1-6.15, chapter 7, 11.1-11.13, 12.1-12.4, 12.8, 12.9.

The Kynch equation

$$u_t + (v(u)u)_x = 0,\ 0 \le x \le 1,\ t \ge 0, \tag{1}$$

is a model for the concentration $u(x,t)$ of particles settling in a closed container under gravity, directed in the positive $x-$direction. The settling velocity $v(u)$ varies with concentration according to

$$v(u) = (1-u)^\beta,$$

where the close packing concentration has been normalized to 1 and β is constant. In this case $\beta = 1.5$. The boundary conditions are

$u(1,t) = 1$	the concentration at the bottom must be close packing
$u(0,t) = 0$	no influx from the top

The task is to write a MATLAB program to find out at what time all particles have settled at the bottom if the initial concentration profile is

$$u_0(x) = \begin{cases} 0 & \text{if } x = 0, \\ 0.9 & \text{if } 0.3 \le x \le 0.5 \\ 0.3 + 0.2x & \text{if } 0 < x < 0.3,\ 0.5 < x < 1 \\ 1 & \text{if } x = 1. \end{cases}.$$

After a while the solution consists of smooth parts connected by shocks and expansion fans.

1. Calculate the characteristic velocity and use it to (a) show that both boundary conditions are needed, and (b) to compute a rule of the thumb maximal length of the time step.

2. It would also be a well-posed problem not to specify any condition at $x = 1$, but the solution would be different, of course. What is the physical interpretation of such a case?

3. Numerical solution

   a) To begin with, write a program for the linear advection equation, $q_t + q_x = 0$, with periodic boundary conditions at $x = 0$ and $x = 1$. Use the high resolution method with numerical flux given by (6.32). See also (6.34) and (6.35). Note that the first two terms in (6.32) are obtained by solving the Riemann problem with piecewise constant initial data. Use $\Delta t/\Delta x$ fixed and run the program until $t = 5$ with different flux-limiter functions. See (6.39). Try at least upwind, Lax-Wendroff, minmod and superbee limiter functions. Use the following initial data:

$$q_1(x,0) = \begin{cases} 1 & \text{if } 0.4 < x < 0.6, \\ 0 & \text{otherwise,} \end{cases}$$

$$q_2(x,0) = \sin(30\pi x)e^{-100(x-0.5)^2}.$$

Compare the numerical solutions with each other and with the exact solution. Comment the performence of the different limiter functions with respect to damping, dispersion, and accuracy. Make sure your method is conservative.

b) Replace the periodic boundary condition with an inflow boundary condition, $q(0,t) = 1$. Run the program and make sure the boundary condition influences the solution in the correct way.

c) Modify your program to solve the non-linear Kynch equation. Show that the upwind flux (12.2) is obtained by solving the non-linear Riemann problem. In the high order correction $\overline{u}$ is replaced by a local wave speed. See (12.22) and (12.7). Remember to implement the boundary conditions. Choose a limiter function and run the program until a time t_0 where at least 4 fronts (shocks and rareefactions) are seen. Hand in a graph with plots of $u(x,t_0)$ for $h = 0.05, 0.025, 0.0125$ (3 curves in the same graph). You should also check the convergence rate.

4. Run the program longer. What happens after long time?